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Distance Education Division
(Degree Program)



Module For:
Operations Research (MGMT 3132)

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Introduction to the Course

Dear Learners, Welcome to the course Operations Research. Operations Research (also referred to Management Science, quantitative methods, quantitative analysis, and decision sciences) is the application of a scientific approach to solving management problems in order to help managers make better decisions. As implied by this definition, management science encompasses a number of mathematically oriented techniques that have been developed within the field of management science or been adapted from other disciplines such as the natural sciences, mathematics, statistics, and engineering. Management science, although rather young, is recognized and established discipline in the field of business administration. The applications of management science/ operations research techniques are wide spread, and they have been frequently credited with increasing the efficiency and productivity of business firms. Operations Research, as one of the quantitative aid to decision-making, offers the decision-maker a method of evaluating every possible alternative (act or course of action) by using various techniques to know the potential outcomes. This is not to say, however, that management decision-making is simply about the application of operations research techniques. In general, while solving a real-life problem, the decision-maker must examine it both from quantitative as well as qualitative perspective. Information about the problem from both these perspectives needs to be brought together and assessed in the context of the problem. Based on some mix of the two sources of information, a decision should be taken by the decision-maker. The study of these methods and how decision-makers use them in the decision process is the essence of operations research approach.

Course Objectives: At the end of this course, you should be able to:

- Understand the meaning of operations research and areas of application;
- Understand linear programming formulation and solution methods;
- Demonstrate post optimality analysis;
- Understand transportation and assignment problems;
- Analyze decision theory approach to decision making;
- Give a general description of PERT/CPM techniques and understand game theory.

Introduction to the Module

Dear Learners! In this module, the overall concepts of operations research, the concepts and methods of solving linear programming are introduced. Each unit begins with brief introduction and objectives of the unit so that students know what is expected from them at end of each unit. In each unit a new concept is supported with the examples. The exercises and activities are given just after discussing issues so that student immediately check themselves to what extent they have understood the subject. The answers to these exercises/activities are provided at the end of each unit. Students are advised not to refer the answers before trying answers the questions by themselves. Summaries are given at the end of each unit to remind students about the main points raised in the unit. Finally, taking in to account the nature of distance education and assuming a high degree of diversity between the learners of distance education a considerable effort has been exerted to ensure that this self-instructional material is easily understandable.

Objective of the module: After completing this module, you should be able to:

- Understand concepts of operations research;
- Formulate and solve linear programming problems;
- Perform development of dual problem from primal problem;
- Demonstrate sensitivity analysis;
- Solve the transportation and assignment problems using different techniques.

UNIT ONE: OVERVIEW OF OPERATIONS RESEARCH

1. Introduction

***Dear Learners!** Operations Research (OR) often referred to as **Management Science** is simply scientific approach to decision-making that seeks to best design and operate a system, usually under condition requiring the allocation of scarce resources. By a system, we mean an organization of interdependent component that work together to accomplish the goal of the system. Management science is a discipline devoted to solving certain managerial problems using quantitative models. Scientific approach to decision-making usually involves the use of one or more **mathematical models**. A mathematical model is a mathematical representation of an actual situation that may be used to make better decisions or simply to understand the actual situation better. Management science/ Operations research use a logical approach to problem solving. The problem is viewed as the focal point of analysis, and quantitative models are the vehicles by which solution are obtained. This quantitative approach is widely employed in areas of application include forecasting, capital budgeting, capacity planning, scheduling, inventory management, project management and production planning. In this first unit, some of the basics of operations research are covered, including the answers to such questions as: what is Origin / evolution of OR? What is operations research? Who uses it? Why use a quantitative approach? What are models, and why are they used? What are Features/Characteristics of OR? What are the Phases of an operations research study? What are the major areas of application?*

Learning Objectives: After completing this unit, you will be able to:

- Identify Definition and Characteristics of Operation Research (OR);
- Explain the evolution of OR;
- Define a model, and how are models used in operations research;
- State Phases of an operations research study;
- Describe some of the important application areas of operations research.

Dear learners, *what do you think of the term operation research?*

1.1. Overview of Operations Research

Operations Research (OR), also known as Operational research, is the application of scientific methods and techniques to problems of decision making and efficiency, especially in the fields of business management and government administration. The terms operations research and management science are often used synonymously. When a distinction is drawn, management science generally implies a closer relationship to the problems of business management. The field of operations research is closely related to Industrial engineering. Industrial engineers typically consider Operations Research (OR) techniques to be a major part of their tool set. Some of the primary tools used by operations researchers are statistics, optimization, probability theory, queuing theory, game theory, graph theory, decision analysis, mathematical modeling and simulation. Because of the computational nature of these fields, OR also has ties to computer science, and operations researchers use both custom-written and off-the-shelf software.

Operations research is distinguished by its frequent use to examine an entire management information system, rather than concentrating only on specific elements (though this is often done as well). An operations researcher faced with a new problem is expected to determine which techniques are most appropriate given the nature of the system, the goals for improvement, and constraints on time and computing power. For this and other reasons, the human element of OR is vital. Like any other tools, OR techniques cannot solve problems by themselves.

1.2. History of Operation Research (OR)

The historical development of Operational Research (OR) is traditionally seen as the succession of several phases: the “heroic times” of the Second World War, the “Golden Age” between the fifties and the sixties during which major theoretical achievements were accompanied by a widespread diffusion of OR techniques in private and public organizations, a “crisis” followed by a “decline” starting with the late sixties, a phase during which OR

groups in firms progressively disappeared while academia became less and less concerned with the applicability of the techniques developed.. In the current phase, the increase in computing power coupled with the birth of related techniques like business intelligence (BI) and business analytic are leading a resurgence of OR. Some say that Charles Babbage (1791-1871) is the "father of operations research" because his research into the cost of transportation and sorting of mail led to England's universal "Penny Post" in 1840, and studies into the dynamical behavior of railway vehicles in defense of the GWR's broad gauge. The modern field of operations research arose during World War II. Modern operations research originated at the Bawdsey Research Station in the UK in 1937 and was the result of an initiative of the station's superintendent, A. P. Rowe. Rowe conceived the idea as a means to analyze and improve the working of the UK's early warning radar system, Chain Home (CH). Initially, he analyzed the operating of the radar equipment and its communication networks, expanding later to include the operating personnel's behavior. This revealed unappreciated limitations of the CH network and allowed remedial action to be taken. Scientists in the United Kingdom including Patrick Blackett, Cecil Gordon, C.H. Waddington, Owen Wansbrough-Jones, Frank Yates, Jacob Bronowski and Freeman Dyson, and in the United States with George Dantzig looked for ways to make better decisions in such areas as logistics and training schedules. After the war it began to be applied to similar problems in industry. During the Second World War close to 1,000 men and women in Britain were engaged in operational research. About 200 operational research bowfins' worked for the British Army.

Patrick Blackett worked for several different organizations during the war. Early in the war while working for the Royal Aircraft Establishment (RAE) he set up a team known as the "Circus" which helped to reduce the number of anti-aircraft artillery rounds needed to shoot down an enemy aircraft from an average of over 20,000 at the start of the Battle of Britain to 4,000 in 1941. In 1941 Blackett moved from the RAE to the Navy, first to the Royal Navy's Coastal Command, in 1941 and then early in 1942 to the Admiralty. Blackett's team at Coastal Command's Operational Research Section (CC-ORS), included, two future Nobel prize winners, and many other people who went on to be preeminent in their fields undertook a number of crucial analyses that aided the war effort. Britain introduced the convoy system to reduce shipping losses, but while the principle of using warships to accompany merchant

ships was generally accepted, it was unclear whether it was better for convoys to be small or large. Convoys travel at the speed of the slowest member, so small convoys can travel faster. It was also argued that small convoys would be harder for German U-boats to detect. On the other hand, large convoys could deploy more warships against an attacker. Blackett's staff showed that the losses suffered by convoys depended largely on the number of escort vessels present, rather than on the overall size of the convoy. Their conclusion, therefore, was that a few large convoys are more defensible than many small ones. While performing an analysis of the methods used by RAF Coastal Command to hunt and destroy submarines, one of the analysts asked what color the aircraft were. As most of them were from Bomber Command they were painted black for nighttime operations. At the suggestion of CC-ORS a test was run to see if that was the best color to camouflage the aircraft for daytime operations in the grey North Atlantic skies. Tests showed that aircraft painted white were on average not spotted until they were 20% closer than those painted black. This change indicated that 30% more submarines would be attacked and sunk for the same number of sightings.

Other work by the CC-ORS indicated that on average if the depth at which aerial delivered depth charges (DC's) was changed from 100 feet to 25 feet, the kill ratios would go up. This was because if a U-boat saw an aircraft only shortly before it arrived over the target then at 100 feet the charges would do no damage, and if it saw the aircraft a long way from the target it had time to alter course under water so the chances of it being within the 20 foot kill zone of the charges was small. It was more efficient to attack those submarines close to the surface whose location was known than those at a greater depth whose position could only be guessed. Before the change from 100 feet to 25 feet 1% of submerged U-boats were sunk and 14% damaged, after the change 7% were sunk and 11% damaged (if caught on the surface the numbers were 11% sunk and 15% damaged). Blackett observed "there can be few cases where such a great operational gain had been obtained by such a small and simple change of tactics. Bomber Command's Operational Research Section (BC-ORS), analyzed a report of a survey carried out by RAF Bomber Command. For the survey, Bomber Command inspected all bombers returning from bombing raids over Germany over a particular period. All damage inflicted by German air defenses was noted and the recommendation was given that armour is added in the most heavily damaged areas. Their suggestion to remove some of

the crew so that an aircraft loss would result in fewer personnel loss was rejected by RAF command. Blackett's team instead made the surprising and counter-intuitive recommendation that the armour be placed in the areas which were completely untouched by damage in the bombers which returned. They reasoned that the survey was biased, since it only included aircraft that returned to Britain. The untouched areas of returning aircraft were probably vital areas, which, if hit, would result in the loss of the aircraft. When Germany organized its air defenses into the Kammhuber Line, it was realized that if the RAF bombers were to fly in a bomber stream they could overwhelm the night fighters who flew in individual cells directed to their targets by ground controllers. It was then a matter of calculating the statistical loss from collisions against the statistical loss from night fighters to calculate how close the bombers should fly to minimize RAF losses.

The "exchange rate" ratio of output to input was a characteristic feature of operations research. By comparing the number of flying hours put in by Allied aircraft to the number of U-boat sightings in a given area, it was possible to redistribute aircraft to more productive patrol areas. Comparison of exchange rates established "effectiveness ratios" useful in planning. The ratio of 60 mines laid per ship sunk was common to several campaigns: German mines in British ports, British mines on German routes, and United States mines in Japanese routes.

Operations research doubled the on-target bomb rate of B-29s bombing Japan from the Marianas Islands by increasing the training ratio from 4 to 10 percent of flying hours; revealed that wolf-packs of three United States submarines were the most effective number to enable all members of the pack to engage targets discovered on their individual patrol stations; revealed that glossy enamel paint was more effective camouflage for night fighters than traditional dull camouflage paint finish, and the smooth paint finish increased airspeed by reducing skin friction. On land, the operational research sections of the Army Operational Research Group (AORG) of the Ministry of Supply (MoS) were landed in Normandy in 1944, and they followed British forces in the advance across Europe. They analyzed, among other topics, the effectiveness of artillery, aerial bombing, and anti-tank shooting.

After World War II from 1962, military operational research in the United Kingdom became known as "operational analysis" (OA) within the UK Ministry of Defense, where OR stands for "Operational Requirement". With expanded techniques and growing awareness, military

OR or OA was no longer limited to only operations, but was extended to encompass equipment procurement, training, logistics and infrastructure.

1.3. Scope of Operations Research

Operations research is wide in scope including the following areas of study: Critical path analysis or project planning: identifying those processes in a complex project which affect the overall duration of the project designing the layout of a factory for efficient flow of materials constructing a telecommunications network at low cost while still guaranteeing quality of service (QoS) or Quality of Experience (QoE) if particular connections become very busy or get damaged road traffic management and 'one way' street allocations i.e. allocation problems. Determining the routes of school buses (or city buses) so that as few buses are needed as possible designing the layout of a computer chip to reduce manufacturing time (therefore reducing cost) managing the flow of raw materials and products in a supply chain based on uncertain demand for the finished products efficient messaging and customer response tactics robotizing or automating human-driven operations processes globalizing operations processes in order to take advantage of cheaper materials, labor, land or other productivity inputs managing freight transportation and delivery systems (Examples: LTL Shipping, inter modal freight transport) scheduling, personnel staffing, manufacturing steps project tasks. Network data traffic: these are known as queuing models or queuing systems sports events and their television coverage blending of raw materials in oil refineries determining optimal prices, in many retail and B2B settings, within the disciplines of pricing science Operations research is also used extensively in government where evidence-based policy is used.

1.4. Applications Areas of Operations Research

➤ Transportation Problem

Goods have to be transported from sources (like factories) to destinations (like warehouses) on a regular basis. The transportation problem deals with minimizing the costs in doing so. Linear programming effectively deals with this problem.

➤ Military Applications

To provide the required protection at the minimum cost, linear programming is used. This technique is useful to cause maximum damage to the enemy with minimum fuel/cost.

➤ Operation of System of Dams

Linear programming is used to find the variations in water storage of dams which generate power, thus maximizing the energy got from the entire system.

➤ Personnel Assignment Problem

If we are given the number of persons, number of jobs and the expected productivity of a particular person on a particular job, linear programming is used to maximize the average productivity of a person.

➤ Other Applications

A traveling sales man can find the shortest routes to save time / fuel cost. The most economic and efficient manner of locating manufacturing plants and distribution centers may be used.

Linear programming may be used for effective and efficient production management and manpower management.

1.5. Nature of Operations Research

“Operations research” and “management science” are terms that are used interchangeably to describe the discipline of using advanced analytical techniques to make better decisions and to solve problems. The procedures of operations research were first formalized by the military. They have been used in wartime to effectively deploy radar, search for enemy submarines, and get supplies to where they are most needed. In peacetime and in private enterprises, operations research is used in planning business ventures and analyzing options by using statistical analysis, data and computer modeling, linear programming, and other mathematical techniques. Large organizations are very complex. They must effectively manage money, materials, equipment, and people. Operations research analysts find better ways to coordinate these elements by applying analytical methods from mathematics, science, and engineering. Analysts often find many possible solutions for meeting the goals of a project. These potential solutions are presented to managers, who choose the course of action that they think best.

Operations research analysts are often involved in top-level strategizing, planning, and forecasting. They help to allocate resources, measure performance, schedule, design production facilities and systems, manage the supply chain, set prices, coordinate transportation and distribution, or analyze large databases.

The duties of the operations research analyst vary according to the structure and management of the organization they are assisting. Some firms centralize operations research in one department; others use operations research in each division. Operations research analysts also may work closely with senior managers to identify and solve a variety of problems. Analysts often have one area of specialization, such as working in the transportation or the financial services industry. Operations research analysts start a project by listening to managers describe a problem. Then, analysts ask questions and formally define the problem. For example, an operations research analyst for an auto manufacturer may be asked to determine the best inventory level for each of the parts needed on a production line and to ascertain the optimal number of windshields to be kept in stock. Too many windshields would be wasteful and expensive, whereas too few could halt production. Analysts would study the problem, breaking it into its components. Then they would gather information from a variety of sources. To determine the optimal inventory, operations research analysts might talk with engineers about production levels, discuss purchasing arrangements with buyers, and examine storage-cost data provided by the accounting department. Relevant information in hand, the analysts determine the most appropriate analytical technique. Techniques used may include a Monte Carlo simulation, linear and nonlinear programming, dynamic programming, queuing and other stochastic-process models, Markov decision processes, econometric methods, data envelopment analysis, neural networks, expert systems, decision analysis, and the analytic hierarchy process. Nearly all of these techniques involve the construction of a mathematical model that attempts to describe the system being studied. So, the problem of the windshields, for example, would be described as a set of equations that try to model real-world conditions. The use of models enables the analyst to explicitly describe the different components and clarify the relationships among them. The descriptions can be altered to examine what may happen to the system under different circumstances. In most cases, a computer program is developed to numerically evaluate the model.

Usually the model chosen is modified and run repeatedly to obtain different solutions. A model for airline flight scheduling, for example, might stipulate such things as connecting cities, the amount of fuel required to fly the routes, projected levels of passenger demand, varying ticket and fuel prices, pilot scheduling, and maintenance costs. By assessing different possible schedules, the analyst is able to determine the best flight schedule consistent with particular assumptions. Based on the results of the analysis, the operations research analyst presents recommendations to managers. The analyst may need to modify and rerun the computer program to consider different assumptions before presenting the final recommendation. Once managers reach a decision, the analyst usually works with others in the organization to ensure the plan's successful implementation.

Summary

***Operations Research (OR)**, also known as Operational research, is the application of scientific methods and techniques to problems of decision making and efficiency, especially in the fields of business management and government administration. The terms operations research and management science are often used synonymously.*

*Management science is a discipline devoted to solving certain managerial problems using quantitative models. Scientific approach to decision-making usually involves the use of one or more **mathematical models**. A mathematical model is a mathematical representation of an actual situation that may be used to make better decisions or simply to understand the actual situation better. Management science/ Operations research use a logical approach to problem solving. The modern field of operations research arose during World War II. Modern operations research originated at the Bawdsey Research Station in the UK in 1937.*

Self Text Exercises 1

Part I: Choose the correct answer & encircle the letter of your choice.

1. A model is –
 - A. Abstraction of reality
 - B. Idealized representation.
 - C. Simple explanation
 - D. A and B
2. A mathematical model usually contain
 - A. Variables
 - B. Constants
 - C. Assumptions
 - D. All of the above
3. One of the following is not the Applications Areas of Operations Research?
 - A. Transportation
 - B. Military
 - C. Personnel Assignment Problem
 - D. Operation system
 - E. None
4. Two of the steps in the management science approach to problem solving are:
 - A. Model construction and model interpretation
 - B. Model construction and model interpretation & analysis
 - C. Finding a solution and testing the model
 - D. Interpret and analyze the model and construct solution.

Part II: Provide brief explanation for each of the following questions.

1. What does it mean operation research?
2. What is the importance of operation research?
3. Discuss the historical development of operation research.

UNIT TWO: LINEAR PROGRAMMING

2. Introduction

Dear Learners, *this is the second unit. The concepts that are going to be discussed in this unit are built on the foundation laid in the first unit. In the previous section, you have been introduced with Operations Research (management science) particularly definitions of operations research, evolution of operations research, features of OR, model and model types and some important ant areas of application. This unit begins with the coverage of linear programming, which is one of the most popular tools of Operations Research (management science). **Linear programming (LP)** model enable users to find optimal solutions to certain problems in which the solutions must satisfy a given set of requirements or constraints. The purpose of this unit is to provide you with an introduction to linear programming models. Emphasis is placed on familiarization with terminology, problem recognition, model formulation, methods of solving Linear Programming Problem, and examples of applications of linear programming.*

Learning Objectives: After completing this unit, you should be able to:

- Define linear programming;
- Explain General structure of LP model and its assumptions;
- Formulate linear programming problems;
- Generate optimal solutions to a LP problem using Graphical approach and simplex algorithm;
- Understand special issues in LP.

2.1. Linear Programming

Dear learners, how are you going to define the term linear programming? Use the space provided below to express your feelings.

2.2. Definition of Linear Programming

In mathematics, linear programming (LP) is a technique for optimization of a linear objective function, subject to linear equality and linear inequality constraints. Informally, linear programming determines the way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model and given some list of requirements represented as linear equations. Linear programming can be applied to various fields of study. Most extensively it is used in business and economic situations, but can also be utilized for some engineering problems. Some industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design.

In mathematics, linear programming (LP) is a technique for optimization of a linear objective function, subject to linear equality and linear inequality constraints. Informally, linear programming determines the way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model and given some list of requirements represented as linear equations. Linear programming can be applied to various fields of study. Most extensively it is used in business and economic situations, but can also be utilized for some engineering problems. Some industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proved useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design.

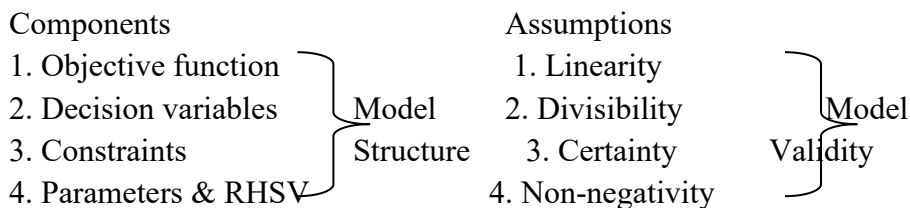
Linear Programming- is an optimization method, which shows how to allocate scarce resources such as money, materials or time and how to do such allocation in the best possible way subject to more than one limiting condition expressed in the form of inequalities and/or equations. It enables users to find optimal solution to certain problems in which the solution must satisfy a given set of requirements or constraints.

Optimization in linear programming implies either maximization (such as profit, revenue, sales, and market share) or minimization (such as cost, time, and distance) a certain objective function. It implies that in LP we cannot max/min two quantities in one model. It involves linearly related multi-variate functions, i.e., functions with more than one independent

variable. The goal in linear programming is to find the best solution given the constraints imposed by the problem; hence the term constrained optimization.

2.3. Linear Programming Models (LPM)

Linear Programming (LP) models are mathematical representations of LP problems. Some LP models have a specialized format, whereas others have a more generalized format. Despite this, LP Models have certain characteristics in common. Knowledge of these characteristics enables us to recognize problems that are amenable to a solution using LP models and to correctly formulate an LP model. The characteristics can be grouped into two categories: **Components** and **Assumptions**. The components relate to the structure of a model, whereas the assumptions reveal the conditions under which the model is valid.



2.4. Components of LP model

1. The Objective Function- is the mathematical or quantitative expression of the objective of the company/model. The objective in problem solving is the criterion by which all decisions are evaluated. In LPMs a single quantifiable objective must be specified by the decision maker. For example, the objective might relate to profits, or costs, or market share, but to only one of these. Moreover, because we are dealing with optimization, the objective will be either maximization or minimization, but not both at a time.

2.The Decision Variables - represent unknown quantities to be resolved for. These decision variables may represent such things as the number of units of different products to be sold, the amount of Birr to be invested in various projects, the number of advertisements to be placed with different media. Since the decision maker has freedom of choice among actions, these decision variables are **controllable variables**.

3. The constraints - are restrictions which define or limit the feasibility of a proposed course of action. They limit the degree to which the objective can be pursued.

Atypical restriction embodies scarce resources (such as labor supply, raw materials, production capacity, machine time, storage space), legal or contractual requirements (e.g. product standards, work standards), or they may reflect other limits based on forecasts, customer orders, company policies etc.

4. Parameters- are fixed values that specify the impact that one unit of each decision variable will have on the objective and on any constraint it pertains to as well as to the numerical value of each constraint.

- The components are the building blocks of an LP model. We can better understand their meaning by examining a simple LP model as follows.

Example:

Maximize: $4X_1 + 7X_2 + 5X_3$ (Profit) _____ objective function

Subject to:

$$2X_1 + 3X_2 + 6X_3 \leq 300 \text{ labor hrs}$$

$$5X_1 + X_2 + 2X_3 \leq 200 \text{ lb raw material A}$$

$$3X_1 + 5X_2 + 2X_3 \leq 360$$

$$X_1 = 30$$

$$X_2 \geq 40$$

$$X_1, X_2, X_3 \geq 0 \rightarrow \text{Non-negativity constraints.}$$

System constraints – involve more than one decision variables.

Individual constraints – involve only one decision variable.

None-negativity constraints specify that no variable will be allowed to take on a negative value. The non-negativity constraints typically apply in an LP model, whether they are explicitly stated or not.

2.5. Assumptions of LP Models

1. **Linearity.** The linearity requirement is that each decision variable has a **linear impact** on the objective function and in each constraint in which it appears. Following the above example, producing one more unit of product 1 adds Br. 4 to the total profit. This is true over the entire range of possible values of X_1 . The same applies to each of the constraints. It is required that the same coefficient (from example, 2 lb. per unit) apply over the entire range of possible value so the decision variable.

2. **Divisibility.** The divisibility requirement pertains to potential values of decision variables. It is assumed that non-integer values are acceptable. For example: 3.5 TV sets/hr would be acceptable → 7 TV sets/2hrs.
3. **Certainty.** The certainty requirement involves two aspects of LP models.
 - i) With respect to model parameters (i.e., the numerical values) – It is assumed that these values are known and constant e.g. in the above example each unit of product 1 requires 2lab hrs is known and remain constant, and also the 300 lab/hr available is deemed to be known and constant.
 - ii) All the relevant constraints identified and represented in the model are as they are.
4. **Non-negativity.** The non-negativity constraint is that negative values of variables are unrealistic and, therefore, will not be considered in any potential solution; only positive values and zero will be allowed.

2.6. Formulating LP Models

Once a problem has been defined, the attention of the analyst shifts to formulating a model. Just as it is important to carefully define a problem, it is important to carefully formulate the model that will be used to solve the problem. If the LP model is ill formulated, ill-structured, it can easily lead to poor decisions.

Formulating linear programming models involves the following steps:

1. Define the problem/problem definition
 - * To determine the # of type 1 and type 2 products to be produced per month so as to maximize the monthly profit given the restrictions.
2. Identify the decision variables or represent unknown quantities
 - * Let X_1 and X_2 be the monthly quantities of Type 1 and type 2 products
3. Determine the objective function
 - * Once the variables have been identified, the objective function can be specified. It is necessary to decide if the problem is a maximization or a minimization problem and the coefficients of each decision variable.

Note: a. The units of all the coefficients in the objective function must be the same.

E.g. If the contribution of type 1 is in terms of Br so does for type 2.

- b. All terms in the objective function must include a variable each term have to have 1 variable.
 - c. All decision variables must be represented in the objective function.
4. Identifying the constraints
- System constraints - more than one variable
 - Individual constraints - one variable
 - Non-negative constraints

Example 1: A firm that assembles computer and computer equipment is about to start production of two new microcomputers. Each type of micro-computer will require assembly time, inspection time and storage space. The amount of each of these resources that can be devoted to the production of microcomputers is limited. The manger of the firm would like to determine the quantity of each microcomputer to produce in order to maximize the profit generated by sales of these microcomputers.

Additional information

In order to develop a suitable model of the problem, the manager has meet with design and manufacturing personnel. As a result of these meetings, the manger has obtained the following information:

	<u>Type 1</u>	<u>Type 2</u>
Profit per unit	Birr 60	Birr 50
Assembly time per unit	4hrs	10hrs
Inspection time per unit	2hrs	1hr
Storage space per unit	3cubic ft	3cubic ft

The manager also has acquired information on the availability of company resources. These weekly amounts are:

<u>Resource</u>	<u>Resource available</u>
Assembly time	100hrs
Inspection time	22hrs
Storage space	39 cubic feet

The manager also met with the firm's marketing manager and learned that demand for the microcomputers was such that whatever combination of these two types of microcomputer is produced, all of the output can be sold.

Required: Formulate the Linear programming model:

Solution:

Step 1: Problem Definition

- To determine the number of two types of microcomputers to be produced (and sold) per week so as to maximize the weekly profit given the restriction.

Step 2: Variable Representation

- Let X_1 and X_2 be the weekly quantities of type 1 and type 2 microcomputers, respectively.

Step 3: Develop the Objective Function

$$\text{Maximize or } Z \text{ max} = 60X_1 + 50X_2$$

Step 4: Constraint Identification

System constraints: $4X_1 + 10X_2 \leq 100\text{hrs}$ Assembly time

$$2X_1 + X_2 \leq 22\text{hrs} \quad \text{inspector time}$$

$$3X_1 + 3X_2 \leq 39 \text{ cubic feet} \quad \text{Storage space}$$

Individual constraint: No

Non-negativity constraint: $X_1, X_2 \geq 0$

In summary, the mathematical model for the microcomputer problem is:

$$Z \text{ max} = 60X_1 + 50X_2$$

$$\text{Subject to: } 4X_1 + 10X_2 \leq 100$$

$$2X_1 + X_2 \leq 22$$

$$3X_1 + 3X_2 \leq 39$$

$$X_1, X_2 \geq 0$$

2. An electronics firm produces three types of switching devices. Each type involves a two-step assembly operation. The assembly times are shown in the following table:

Assembly time per Unit (in minutes).	Section #1	Section #2
Model A	2.5	3.0
Model B	1.8	1.6
Model C	2.0	2.2

Each workstation has a daily working time of 7.5 hrs. The manager wants to obtain the greatest possible profit during the next five working days. Model A yields a profit of Birr 8.25 per unit, Model B a profit of Birr 7.50 per unit and Model C a profit of Birr 7.80 per unit. Assume that the firm can sell all it produces during this time, but it must fill outstanding orders for 20 units of each model type.

Required: Formulate the linear programming model of this problem.

Solution:

Step 1. Problem definition

To determine the number of three types of switching devices to be produced and sold for the next 5 working days so as to maximize the 5 days profit.

Step 2. Variable representation

Let X_1 , X_2 and X_3 be the number of Model A, B and C switching devices respectively, to be produced and sold.

Step 3. Develop objective function

$$Z \text{ max: } 8.25X_1 + 7.50X_2 + 7.80X_3$$

Step 4. Constraint identification

$$\begin{array}{ll} 2.5X_1 + 1.8X_2 + 2.0X_3 \leq 2250 \text{ minutes} & \text{Ass. time station 1} \\ 3.0X_1 + 1.6X_2 + 2.2X_3 \leq 2250 \text{ minutes} & \text{Ass. time station 2} \end{array} \left. \vphantom{\begin{array}{l} 2.5X_1 + 1.8X_2 + 2.0X_3 \leq 2250 \text{ minutes} \\ 3.0X_1 + 1.6X_2 + 2.2X_3 \leq 2250 \text{ minutes} \end{array}} \right\} \begin{array}{l} \text{System} \\ \text{constraints} \end{array}$$

$$\begin{array}{ll} X_1 \geq 20 & \text{Model A} \\ X_2 \geq 20 & \text{Model B Individual constraint} \\ X_3 \geq 20 & \text{Model C} \end{array} \left. \vphantom{\begin{array}{l} X_1 \geq 20 \\ X_2 \geq 20 \\ X_3 \geq 20 \end{array}} \right\}$$

$$X_1, X_2, X_3 \geq 0 \quad \text{Non-negativity}$$

In summary:

$$\begin{array}{ll} Z_{\text{max}}: 8.25X_1 + 7.50X_2 + 7.80X_3 & \\ : 2.5X_1 + 1.8X_2 + 2.0X_3 \leq 2250 & \text{minutes} \\ 3.0X_1 + 1.6X_2 + 2.2X_3 \leq 2250 & \text{minutes} \\ X_1 \geq 20 & \text{model A} \\ X_2 \geq 20 & \text{model B} \\ X_3 \geq 20 & \text{model C} \end{array}$$

$$X_1, X_2, X_3 \geq 0 \quad \text{non-negativity}$$

3. A diet is to include at least 140 mgs of vitamin A and at least 145 Mgs of vitamin B. These requirements are to be obtained from two types of foods: Type 1 and Type 2. Type 1 food contains 10Mgs of vitamin A and 20mgs of vitamin B per pound. Type 2 food contains 30mgs of vitamin A and 15 mgs of vitamin B per pound. If type 1 and 2 foods cost Birr 5 and Birr 8 per pound respectively, how many pounds of each type should be purchased to satisfy the requirements at a minimum cost?

<u>Foods</u>	<u>Vitamins</u>	
	<u>A</u>	<u>B</u>
Type 1	10	20
Type 2	30	15

Solution:

Step 1. Problem definition

To determine the pounds of the two types of foods to be purchased to make the diet at a minimum possible cost within the requirements.

Step 2. Variable representation

Let X_1 and X_2 be the number of pounds of type 1 and type 2 foods to be purchased, respectively.

Step 3. Objective function

$$Z_{\min}: 5X_1 + 8X_2$$

4. Constraints

$$\begin{aligned} 10X_1 + 30X_2 &\geq 140 \\ 20X_1 + 15X_2 &\geq 145 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{System constraints}$$

$$X_1, X_2 \geq 0 \quad \text{non-negativity constraints.}$$

4. A farm consists of 600 hectares of land of which 500 hectares will be planted with corn, barley and wheat, according to these conditions.

- (1) At least half of the planted hectare should be in corn.
- (2) No more than 200 hectares should be barley.
- (3) The ratio of corn to wheat planted should be 2:1

It costs Birr 20 per hectare to plant corn, Birr 15 per hectare to plant barley and Birr 12 per hectare to plant wheat.

- a. Formulate this problem as an LP model that will minimize planting cost while achieving the specified conditions.

Solution:

Step 1. Problem definition

To determine the number of hectares of land to be planted with corn, barley and wheat at a minimum possible cost meeting the requirements.

Step 2. Decision variable representation

Let X_1 be the number of hectares of land to be planted with corn, X_2 be the number of hectares of land to be planted with barley, and X_3 be the number of hectares of land to be planted with wheat.

Step 3. Objective function

$$Z_{\min} = 20X_1 + 15X_2 + 12X_3$$

Step 4. Constraints

$$X_1 + X_2 + X_3 = 500$$

$$X_1 \geq 250$$

$$X_2 \leq 200$$

$$X_1 - 2X_3 = 0$$

$$X_1, X_2, X_3 \geq 0$$

In summary

$$Z_{\min}: 20X_1 + 15X_2 + 12X_3$$

$$\text{S.t. } X_1 + X_2 + X_3 = 500$$

$$X_1 - 2X_3 = 0$$

$$X_1 \geq 250$$

$$X_2 \leq 200$$

$$X_1, X_2, X_3 \geq 0$$

2.7. Solution Approaches to Linear Programming Problems

There are two approaches to solve linear programming problems:

1. The Graphic solution method
2. The Algebraic solution/ simplex algorithm method

A. The Graphic Solution Method

It is a relatively straightforward method for determining the optimal solution to certain linear programming problems. It gives as a clear picture. This method can be used only to solve problems that involve two decision variables. However, most linear programming applications involve situations that have more than two decision variables, so the graphic approach is not used to solve them.

E.g.: 1. Solving the micro-computer problem with graphic approach

$$Z \text{ max} = 60X_1 + 50X_2$$

$$\text{S.t. } 4X_1 + 10X_2 \leq 100$$

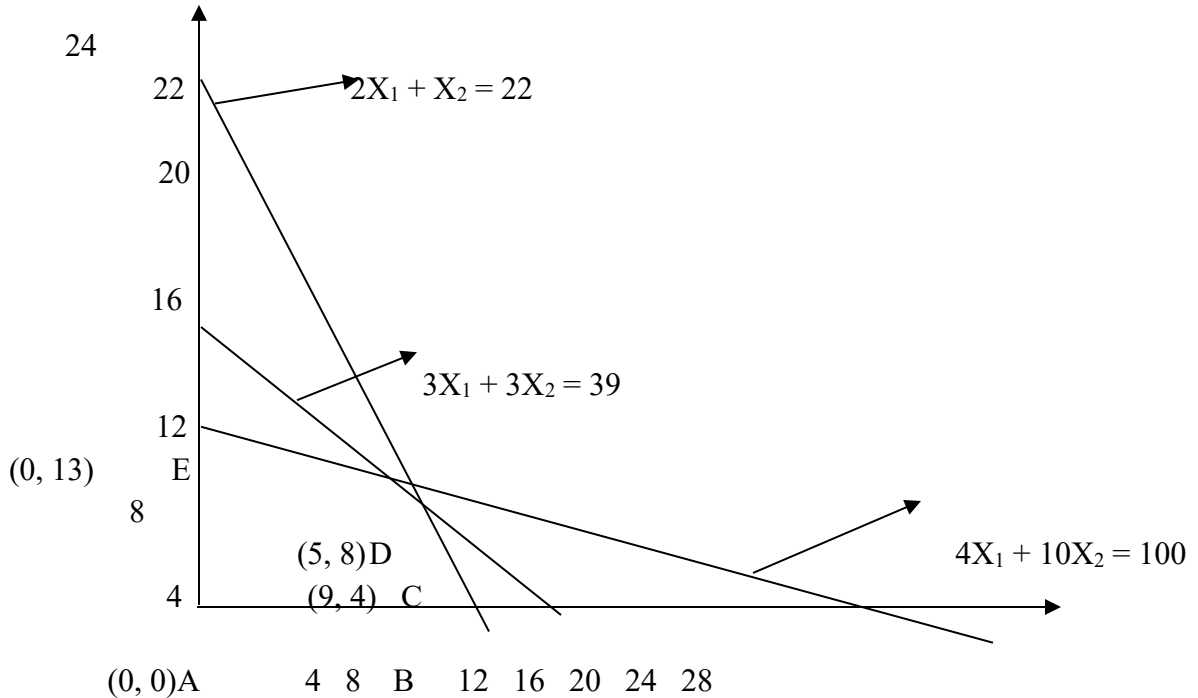
$$2X_1 + X_2 \leq 22$$

$$3X_1 + 3X_2 \leq 39$$

$$X_1, X_2 \geq 0$$

Steps:

1. Plot each of the constraints and identify its region – make linear inequalities linear equations.
2. Identify the common region, which is an area that contains all of the points that satisfy the entire set of constraints.
3. Determine the Optimal solution- identify the point which leads to maximum benefit or minimum cost.



To identify the maximum (minimum) value we use the *corner point* approach or the *extreme point* approach. The corner point/extreme point approach has one theorem: It states that;

For problems that have optimal solutions, a solution will occur at an extreme, or corner point. Thus, if a problem has a single optimal solution, it will occur at a corner point. If it has multiple optimal solutions, at least one will occur at a corner point. Consequently, in searching for an optimal solution to a problem, we need only consider the extreme points because one of those must be optimal. Further, by determining the value of the objective function at each corner point, we could identify the optimal solution by selecting the corner point that has the best value (i.e., maximum or minimum, depending on the optimization case) at the objective function.

Determine the values of the decision variables at each corner point. Sometimes, this can be done by inspection (observation) and sometimes by simultaneous equation. Substitute the value of the decision variables at each corner point.

After all corner points have been so evaluated, select the one with the highest or lowest value depending on the optimization case.

Points	Coordinates X ₁ X ₂		How Determined	Value of Objective function $Z = 60X_1 + 50X_2$
A	0	0	Observation	Birr 0
B	11	0	Observation	Birr 660
C	9	4	Simultaneous equations	Birr 740
D	5	8	Simultaneous equations	Birr 700
E	0	10	Observation	Birr 500

Basic solution:

$$X_1 = 9, \quad X_2 = 4, \quad Z = \text{Birr } 740$$

After we have got the optimal solution, we have to substitute the value of the decision variables into the constraints and check whether all the resources available were used or not. If there is an unused resource we can use it for any other purpose. The amount of unused resources is known as *SLACK*-the amount of the scarce resource that is unused by a given solution.

The slack can range from zero, for a case in which all of a particular resource is used, to the original amount of the resource that was available (i.e., none of it is used).

Computing the amount of slack

Constraint	Amount used with X ₁ = 9 and X ₂ = 4	Originally available	Amount of slack (available – Used)
Assembly time	$4(9) + 10(4) = 76$	100 hrs	$100 - 76 = 24 \text{ hrs}$
Inspection time	$2(9) + 1(4) = 22$	22 hrs	$22 - 22 = 0 \text{ hr}$
Storage space	$3(9) + 3(4) = 39$	39 cubic ft	$39 - 39 = 0 \text{ cubic ft}$

Constraints that have no slack are some time referred to as binding constraints since they limit or bind the solution. In the above case, inspection time and storage space are binding constraints; while assembly time has slack.

Knowledge of unused capacity can be useful for planning. A manager may be able to use the assembly time for other products, or, perhaps to schedule equipment maintenance, safety seminars, training sessions or other activities.

Interpretation: The Company is advised to produce 9 units of type 1 microcomputers and 4 units of type 2 microcomputers per week to maximize his weekly profit to Birr 740; and in do so the company would be left with unused resource of 24-assembly hrs that can be used for other purposes.

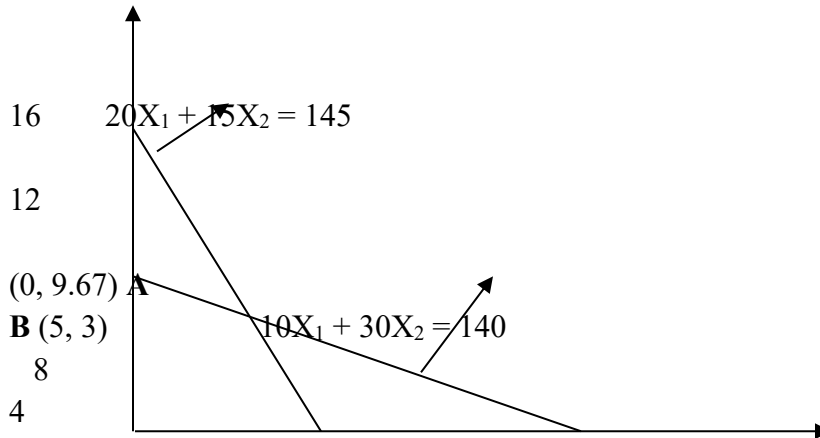
2. Solving the diet problem with graphic approach

$$C_{\min}: 5X_1 + 8X_2$$

$$10X_1 + 30X_2 \geq 140$$

$$20X_1 + 15X_2 \geq 145$$

$$X_1, X_2 \geq 0$$



$$812 \quad C(14, 0) \quad 1620$$

Basic solution: $X_1 = 5$ pounds

$X_2 = 3$ pounds

$C = \text{Birr } 49$

Interpretation: To make the diet at the minimum cost of Birr 49 we have to purchase 5

Points	Coordinates		How Determined	Value of the objective function
	X_1	X_2		$Z = 5X_1 + 8X_2$
A	0	9.67	Observation	Birr 77.30
B	5	3	Simultaneous equations	Birr 49
C	14	0	Observation	Birr 70

pounds of Type 1 food and 3 pounds Type 2 food.

If there is a difference between the minimum required amount and the optimal solution, we call the difference *surplus*: That is, Surplus is the amount by which the optimal solution causes a $\geq$ constraint to exceed the required minimum amount. It can be determined in the same way that slack can: substitute the optimal values of the decision variables into the left side of the constraint and solve. The difference between the resulting value and the original right-hand side amount is the amount of surplus. Surplus can potentially occur in a $\geq$ constraint.

B. The Simplex Algorithm/Algebraic Solution Method

The simplex method is an iterative technique that begins with a feasible solution that is not optimal, but serves as a starting point. Through algebraic manipulation, the solution is improved until no further improvement is possible (i.e., until the optimal solution has been identified). Each iteration moves one step closer to the optimal solution. In each iteration, one variable that is not in the solution is added to the solution and one variable that is in the solution is removed from the solution in order to keep the number of variables in the basis equal to the number of constraints.

The optimal solution to a linear programming model will occur at an extreme point of the feasible solution space. This is true even if a model involves more than two variables; optimal solutions will occur at these points. Extreme points represent intersections of constraints. Of course, not every solution will result in an extreme point of the feasible solution space; some will be outside of the feasible solution space. Hence, not every solution will be a feasible solution. Solutions which represent *intersections of constraints* are called **basic solutions**; those which also satisfy all of the constraints, including the non-negativity constraints, are called **basic feasible solutions**. The simplex method is an algebraic procedure for systematically examining basic feasible solutions. If an optimal solution exists, the simplex method will identify it.

The simplex procedure for a maximization problem with all $\leq$ constraints consists of the following steps.

1. Write the LPM in a standard form: when all of the constraints are written as equalities, the linear program is said to be in standard form. We convert the LPM into a standard form by applying the slack variables, S , which carries a subscript that denotes which constraint it applies to. For example, S_1 refers to the amount of slack in the first constraint, S_2 to the amount of slack in the second constraint, and so on. When slack variables are introduced to the constraints, they are no longer inequalities because the slack variable accounts for any difference between the left and right-hand sides of an expression. Hence, once slack variables are added to the constraints, they become equalities. Furthermore, every variable in a model must be represented in the objective function. However, since slack does not provide any real contribution to the objective, each slack variable is assigned a coefficient of zero in the objective function.

Slack = Requirement – Production, surplus = Production – Requirement

Taking the microcomputer problem its standard form is as follows:

$$\begin{array}{ll} Z \text{ max} = 60X_1 + 50X_2 & Z \text{ max} = 60X_1 + 50X_2 + 0S_1 + S_2 + 0S_3 \\ : 4X_1 + 10X_2 \leq 100 & : 4X_1 + 10X_2 + S_1 = 100 \\ 2X_1 + X_2 \leq 22 & 2X_1 + X_2 + S_2 = 22 \end{array}$$

$$3X_1 + 3X_2 \leq 39$$

$$X_1, X_2 \geq 0$$

$$3X_1 + 3X_2 + S_3 = 39$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

2. Develop the initial tableau: the initial tableau always represents the "Do Nothing" strategy, so that the decision variables are initially non-basic.

- List the variables across the top of the table and write the objective function coefficient of each variable just above it.
- There should be one row in the body of the table for each constraint. List the slack variables in the basis column, one per row.
- In the C_j column, enter the objective function coefficient of zero for each slack variable. (C_j - coefficient of variable j in the objective function)
- Compute values for row Z_j
- Compute values for $C_j - Z_j$.

Sol/n basis	C_j	60 X_1	50 X_2	0 S_1	0 S_2	0 S_3	RHSV	$\theta_j = b_j/x_j (a_{ij})$
S1	0	4	10	1	0	0	100	100/4 = 25
S2	0	2*	1	0	1	0	22	22/2 = 11
S3	0	3	3	0	0	1	39	39/3 = 13
	Z_j	0	0	0	0	0	0	
	$C_j - Z_j$	60	50	0	0	0	0	

↑ Entering variable
↓ Pivot column
→ Leaving variable

↘ Pivot row

*** Pivot Element**

- Develop subsequent tableaus
 - Identify the entering variable - a variable that has a largest positive value is the $C_j - Z_j$ row.
 - Identify the leaving variable - Using the constraint coefficients or substitution rates in the entering variable column divide each one into the corresponding quantity value. However do not divide by a zero or negative value. The smallest non-negative ratio that results indicate which variable will leave the solution.
- Find unique vectors for the new basic variable using row operations on the pivot element.

Sol/n basis	C_j	60 X_1	50 X_2	0 S_1	0 S_2	0 S_3	RHSV	$\theta_j = b_j/x_j (a_{ij})$
S1	0	0	8	1	-2	0	56	56/8 = 7
X1	60	1	1/2	0	1/2	0	11	11/.5 = 22
S3	0	0	3/2	0	-3/2	1	6	6/1.5 = 4
	Z_j	60	30	0	30	0	660	
	$C_j - Z_j$	0	20	0	-30	0	0	

↑ Entering Variable
→ Leaving variable

Sol/n basis	C _j	60 X ₁	50 X ₂	0 S ₁	0 S ₂	0 S ₃	RHSV	Ø _j = b _j /x _j (a _{ij})
S1	0	0	0	1	6	-16/3	24	
X1	60	1	0	0	1	-1/3	9	
X2	50	0	1	0	-1	2/3	4	
	Z _j	60	50	0	10	40/3	740	
	C _j -Z _j	0	0	0	-10	-40/3		

Optimal solution: X1 = 9, X2 = 4, S1 = 24 hrs, Z = Birr 740

5. Compute the C_j – Z_j row
6. If all C_j – Z_j values are zeros and negatives you have reached optimality.
7. If this is not the case (step 6), rehear step 2to5 until you get optimal solution.

“A simplex solution is a maximization problem is optimal if the C_j – Z_j row consists entirely of zeros and negative numbers (i.e., there are no positive values in the bottom row).”

Note: The variables in solution all have unit vectors in their respective columns for the constraint equations. Further, note that a zero appears in row c - z in every column whose variable is in solution, indicating that its maximum contribution to the objective function has been realized.

Example 2

A manufacturer of lawn and garden equipment makes two basic types of lawn mowers: a push-type and a self-propelled model. The push-type requires 9 minutes to assemble and 2 minutes to package; the self-propelled mower requires 12 minutes to assemble and 6 minutes to package. Each type has an engine. The company has 12 hrs of assembly time available, 75 engines, and 5hrs of packing time. Profits are Birr 70 for the self-propelled models and Birr 45 for the push-type mower per unit.

Required:

1. Formulate the linear programming models for this problem.
2. Determine how many mower of each type to make in order to maximize the total profit (use the simplex procedure).

Solution:

1. a) To determine how many units of each type of mowers to produce so as to maximize profit.

b) Let X_1 - be push type mower.

X_2 - be self-propelled mower.

c) Determine the objective function

$$Z_{\max} = 45X_1 + 70X_2$$

d) Identify constraints

$$9X_1 + 12X_2 \leq 720 \text{ minutes} \quad \text{Assembly time}$$

$$2X_1 + 6X_2 \leq 300 \text{ minutes} \quad \text{packing time}$$

$$X_1 + X_2 \leq 75 \text{ engines} \quad \text{Engines}$$

$$X_1, X_2 \geq 0$$

In summary:

$$Z_{\max} = 45X_1 + 70X_2$$

$$: 9X_1 + 12X_2 \leq 720$$

$$2X_1 + 6X_2 \leq 300$$

$$X_1 + X_2 \leq 75$$

$$X_1, X_2 \geq 0$$

2. a. Write the LPM in a standard form

$$Z_{\max} = 45X_1 + 70X_2 + 0S_1 + 0S_2 + 0S_3$$

$$: 9X_1 + 12X_2 + S_1 = 720$$

$$2X_1 + 6X_2 + S_2 = 300$$

$$X_1 + X_2 + S_3 = 75$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

a. Develop the initial tableau – in LP matrices are commonly called tableaus

Sol/n basis	Cj	45 X_1	70 X_2	0 S_1	0 S_2	0 S_3	RHSV	$\theta_j = b_j/x_j (a_{ij})$
S1	0	9	12	1	0	0	720	720/12 = 60
S2	0	2	6	0	1	0	300	300/6 = 50 →
S3	0	1	1	0	0	1	75	75/1 = 75
	Zj	0	0	0	0	0	0	
	Cj-Zj	45	70	0	0	0		

Leaving
variable

Entering variable

b. Develop the subsequent tableaus

-Identify the entering variable

-Identify the leaving variable

Sol/n basis	C _j	45 X ₁	70 X ₂	0 S ₁	0 S ₂	0 S ₃	RHSV	Ø _j = b _j /x _j (a _{ij})	
S1	0	5	0	1	-2	0	120	120/5 = 24	→ Leaving variable
X2	70	1/3	1	0	1/6	0	50	50/.333 = 150	
S3	0	2/3	1	0	-1/6	1	25	25/.666 = 75	
	Z _j	70/3	70	0	70/6	0	3500		
	C _j -Z _j	65/3	0	0	-70/6	0			

↑
Entering variable

Sol/n basis	C _j	45 X ₁	70 X ₂	0 S ₁	0 S ₂	0 S ₃	RHSV	Ø _j = b _j /x _j (a _{ij})
X1	45	1	0	1/5	-2/5	0	24	
X2	70	0	1	-1/15	3/10	0	42	
S3	0	0	0	-2/15	1/10	1	9	
	Z _j	45	70	13/3	3	0	4020	
	C _j -Z _j	0	0	-13/3	-3	0		

Optimal solutions: X₁ = 24 units

X₂ = 42 units

S₃ = 9 engines Z = Birr 4020

Interpretation: The Company is advised to produce 24 units of push type mowers and 42 units of self-propelled mowers so as to realize a profit of Birr 4020. In doing so the company would be left with unused resource of 9 engines which can be used for other purposes.

1. A firm produces products A, B, and C, each of which passes through assembly and inspection departments. The number of person hours required by a unit of each product in each department is given in the following table.

Person hours per unit of product

	Product A	Product B	Product C
Assembly	2	4	2
Inspection	3	2	1

During a given week, the assembly and inspection departments have available at most 1500 and 1200 person-hours, respectively. if the unit profits for products A, B, and C are Birr 50, Birr 40, and Birr 60, respectively, determine the number of units of each product that should be produced in order to maximize the total profit and satisfy the constraints of the problem.

Answer: 0 unit of product A, 0 unit of product B, 750 units of product C, unused inspection time of 450 hours, and a maximum profit, Z ,of Birr 45,000.

4. The state chairman of a political party must allocate an advertising budget of birr 3,000,000 among three media: radio, television, and newspapers. The expected number of votes gained per birr spent on each advertising medium is given below.

Expected votes per Birr spent		
Radio	Television	Newspapers
3	5	2

Since these data are valid with in the limited amounts spent on each medium, the chairman has imposed the following restrictions:

- ✍ No more than Birr 500,000 may be spent on radio ads.
- ✍ No more than Birr 1,200,000 may be spent on television ads.
- ✍ No more than Birr 2,400,000 may be spent on television and newspaper ads combined.

How much should be spent on each medium in order to maximize the expected number of votes gained?

Answer: Birr 500,000 should be spent on radio ads.

Birr 1,200,000 should be spent on television ads.

Birr 1,200,000 should be spent on newspaper ads.

Slack in the budget constraint is Birr 100,000.

Z = 9,900,000 is the maximum expected number of votes gained.

2.8. Minimization Linear Programming Problems

2.8.1. Big M-method /Charnes Penalty Method/

The Big-M method is a technique, which is used in removing artificial variables from the basis. In this method; we assign coefficients to artificial variables, undesirable from the objective function point of view. If objective function Z is to be minimized, then a very large positive price (called **penalty**) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative cost (also called **penalty**) is assigned to each of these variables. Following are the characteristics of Big-M Method:

- a. High penalty cost (profit) is assumed as M
- b. M as a coefficient is assigned to artificial variable **A** in the objective function Z.
- c. Big-M method can be applied to minimization as well as maximization problems with the following distinctions:

i. Minimization problems

- Assign $+M$ as coefficient of artificial variable A in the objective function Z of the minimization problem.

ii. Maximization problems:

- Here $-M$ is assigned as coefficient of artificial variable A in the objective function Z of the maximization problem.

d. Coefficient of S (slack/surplus) takes zero values in the objective function Z

e. For minimization problem, the incoming variable corresponds to the **highest negative** value of $C_j - Z_j$.

f. The solution is optimal when there is no negative value of $C_j - Z_j$. (For minimization LPP case)

The various steps involved in using simplex method for minimization problems are:

Step 1: Formulate the linear programming model, and express the mathematical model of LP problem in the standard form by introducing surplus and artificial variables in the left hand side of the constraints. Assign a 0 (zero) and $+M$ as coefficient for surplus and artificial variables respectively in the objective function. M is considered a very large number so as to finally drive out the artificial variables out of basic solution.

Step 2: Next, an initial solution is set up. Just to initiate the solution procedure, the initial basic feasible solution is obtained by assigning zero value to decision variables. This solution is now summarized in the initial simplex table. Complete the initial simplex table by adding two final rows Z , and $C_j - Z_j$. These two rows help us to know whether the current solution is optimum or not.

Step 3: Now; we test for optimality of the solution. If all the entries of $C_j - Z_j$ row are positive, then the solution is optimum. However, this situation may come after a number of iterations. But if at least one of the $C_j - Z_j$ values is less than zero, the current solution can be further improved by removing one basic variable from the basis and replacing it by some non-basic one.

Step 4: (i) Determine the variable to enter the basic solution. To do this, we identify the column with the largest negative value in the $C_j - Z_j$ row of the table.

(ii) Next we determine the departing variable from the basic solution. If an artificial variable goes out of solution, then we discard it totally and even this variable may not form part of further iterations. Same procedure, as in maximization case, is employed to determine the departing variable.

Step 5: We update the new solution now. We evaluate the entries for next simplex table in exactly the same manner as was discussed earlier in the maximization case.

Step 6: Step (3 - 5) are repeated until an optimum solution is obtained.

So the following are the essential things to observe in solving for minimization problems:

- The entering variable is the one with the largest negative value in the $C_j - Z_j$ row while the leaving variable is the one with the smallest non-negative ratio.
- The optimal solution is obtained when the $C_j - z_j$ row contains entirely zeros and positive values.

Example: Assume the following minimization problem.

$$\begin{aligned} \text{Min} \quad & Z = 7X_1 + 9X_2 \\ \text{Subject to} \quad & 3X_1 + 6X_2 \geq 36 \\ & 8X_1 + 4X_2 \geq 64 \\ & X_1, X_2 \geq 0 \end{aligned}$$

We introduce both surplus and artificial variables into both constraints as follows.

$$\begin{aligned} \text{Min} \quad & Z = 7X_1 + 9X_2 + 0S_1 + 0S_2 + MA_1 + MA_2 \\ \text{Subject to} \quad & 3X_1 + 6X_2 - S_1 + A_1 = 36 \\ & 8X_1 + 4X_2 - S_2 + A_2 = 64 \\ & X_1, X_2 \geq 0 \end{aligned}$$

So the subsequent tableaus for this problem are shown below. To remain in these tableaus is in transforming from one tableau to another, we perform elementary row operations to obtain the unit vector in the pivot column for the entering variable not the solution.

Initial Simplex Tableau

C _j		7	9	0	0	M	M	Quantity
Basic V.		X ₁	X ₂	S ₁	S ₃	A ₁	A ₂	
A ₁	M	3	6	-1	0	1	0	36
A ₂	M	8	4	0	-1	0	1	64
Z _j		11M	10M	-M	-M	M	M	100M
C _j -Z _j		7-11M	9-10M	M	M	0	0	

Second Simplex Tableau

C _j		7	9	0	0	M	Quantity
Basic V.		X ₁	X ₂	S ₁	S ₂	A ₁	
A ₂	M	0	9/2	-1	3/8	1	12
X ₁	7	1	1/2	0	-1/8	0	8
Z _j		7	7/2+9/2M	-M	3/8M-7/8	M	56+12M
C _j -Z _j		0	11/2-9/2M	M	7/8-3/8M	0	

Third Simplex Tableau

C _j		7	9	0	0	Quantity
Basic V.		X ₁	X ₂	S ₁	S ₂	
X ₂	9	0	1	-2/9	1/2	8/3
X ₁	7	1	0	1/9	-1/6	20/3
Z _j		7	9	-11/9	-5/12	212/3
C _j -Z _j		0	0	11/9	5/12	

The third tableau represents a final tableau since it is the optimal solution with entirely zeros and non-negative values in the C_j-Z_j row.

Therefore, the optimal solution is: X₁ = 20/3 and X₂ = 8/3 and value of objective function is 212/3.

Summary

Types of constraint	Extra variables to be added	Coefficient of extra variables in the objective function		Presence of variables in the initial solution mix
		Max Z	Min Z	
$\leq$	Add only slack variable	0	0	Yes
$\geq$	Subtract surplus variable and	0	0	No
	Add artificial variable	-M	+M	Yes
=	Add artificial variable	-M	+M	Yes

2.9. Some special Issues in LP

2.9.1. Redundant Constraint

If a constraint when plotted on a graph doesn't form part of the boundary making the feasible region of the problem that constraint is said to be redundant.

Example:

A firm is engaged in producing two products A and B .Each unit of product A requires 2Kg of raw material and 4 labor-hours for processing. Whereas each unit of product B requires 3Kg of raw materials and 3hours of labor. Every unit of product A requires 4 hrs for packaging where as B needs 3.5hrs. Every week the firm has availability of 60Kg of raw material, 96 labor-hours and 105 hrs in the packaging department.1 unit of product A sold yields \$40 profit and 1 unit of B sold yields \$35 profit.

Required:

- Formulate this problem as a LPM
- Find the optimal solution

Solution

<u>Resources</u>	<u>Products</u>		<u>Resource available</u>		<u>per week</u>
			<u>A</u>	<u>B</u>	
Raw materials (Kg)	2	3		60	
Labor (hr)	4	3		96	
Packaging (hr)	4	3.5		105	
Profit per unit	<u>\$40</u>	<u>\$35</u>			

Let X_1 = The N° of units of product A produced per week

X_2 = The N° of units of product B produced per week

a. LP Model

$$\text{Max. } Z = 40X_1 + 35X_2$$

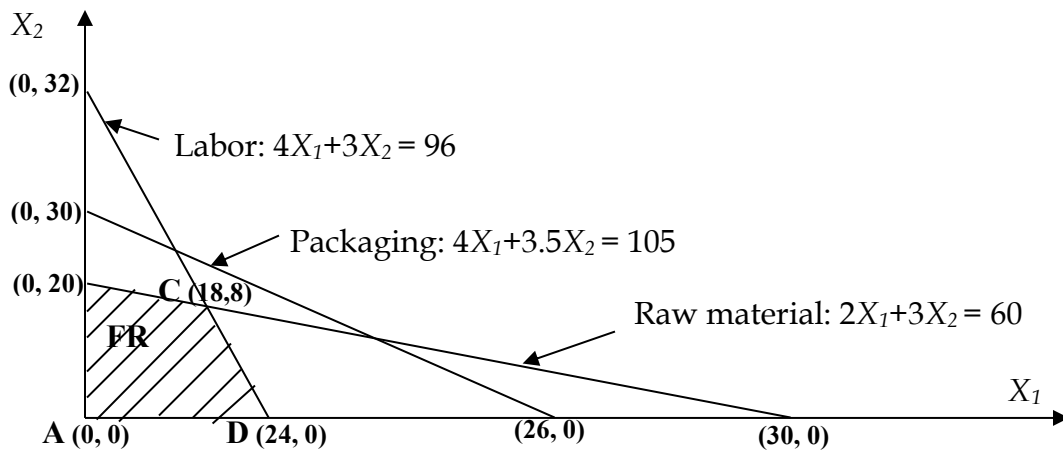
St :

$$2X_1 + 3X_2 \leq 60$$

$$4X_1 + 3X_2 \leq 96$$

$$4X_1 + 3.5X_2 \leq 105$$

$$X_1, X_2 \geq 0$$



❖ The packaging hr is redundant.

<u>Corners</u>	<u>Coordinates</u>	<u>Max $Z=40X_1 + 35X_2$</u>
A	(0, 0)	0
B	(0, 20)	700
C	(18, 8)	1000
D	(24, 0)	960

$$X_1=18, X_2=8 \text{ and Min } Z=1000$$

Interpretation:

The company should produce and sale 18 units of product A and 8 units of product B per week so as to get a maximum profit of 1000.

- ❖ By this production plan the entire raw material will be consumed.

$$2X_1+3X_2 \leq 60$$

$$2(18) + 3(8) = 60$$

$$\underline{60 = 60} \implies \text{No idle or unused raw material}$$

- ❖ $4X_1+3X_2 \leq 96$

$$4(18) + 3(8) \leq 96$$

$$\underline{96 = 96} \implies \text{the entire labor hour will be consumed}$$

- ❖ $4X_1+3.5X_2 \leq 105$

$$100 \leq 105 \implies \text{There is to be idle or unused capacity of 5hrs in the packaging department.}$$

Note:

The packaging hour's constraint does not form part of the boundary making the feasible region. Thus, this constraint is of no consequence and is therefore, redundant. The **inclusion** or **exclusion** of a redundant constraint does not affect the **optimal solution** of the problem.

2.9.2. Multiple optimal Solutions /Alternative optimal solutions/

The same maximum value of the objective function might be possible with a number of different combinations of values of the decision variables. This occurs because the objective function is parallel to a binding constraint. With simplex method this condition can be detected by examining the $C_j - Z_j$ row of the final tableau. If a zero appears in the column of a non-basic variable (i.e., a variable that is not in solution), it can be concluded that an alternate solution exists.

-This is a situation where by a LPP has more than one optimal solution.

Multiple optimal Solutions will be found if two corners give optimal solution, then the line segment joining these points will be the solution.

$\implies$ We have unlimited number of optimal solution without increasing or decreasing the objective function.

E.g. $Z = 60X_1 + 30X_2$

$$4X_1 + 10X_2 \leq 100$$

$$2X_1 + X_2 \leq 22$$

$$3X_1 + 3X_2 \leq 39$$

$$X_1, X_2 \geq 0$$

The other optimal corner point can be determined by entering the non-basic variable with the $C - Z$ equal to zero and, then, finding the leaving variable in the usual way.

Example:

The information given below is for the products A and B.

<i>Department</i>	<i>Machine hours per week</i>		<i>Maximum available per week</i>
	<i>Product A</i>	<i>Product B</i>	
Cutting	3	6	900
Assembly	1	1	200
<i>Profit per unit</i>	<u><i>\$8</i></u>	<u><i>\$16</i></u>	

Assume that the company has a marketing constraint on selling products B and therefore it can sale a maximum of 125units of this product.

Required:

- Formulate the LPM of this problem
- Find the optimal solution

Solution:

Let X_1 =The N° of units of product A produced per week

X_2 =The N° of units of product B produced per week

- The LP Model of the problem is:

$$\text{Max. } Z = 8X_1 + 16X_2$$

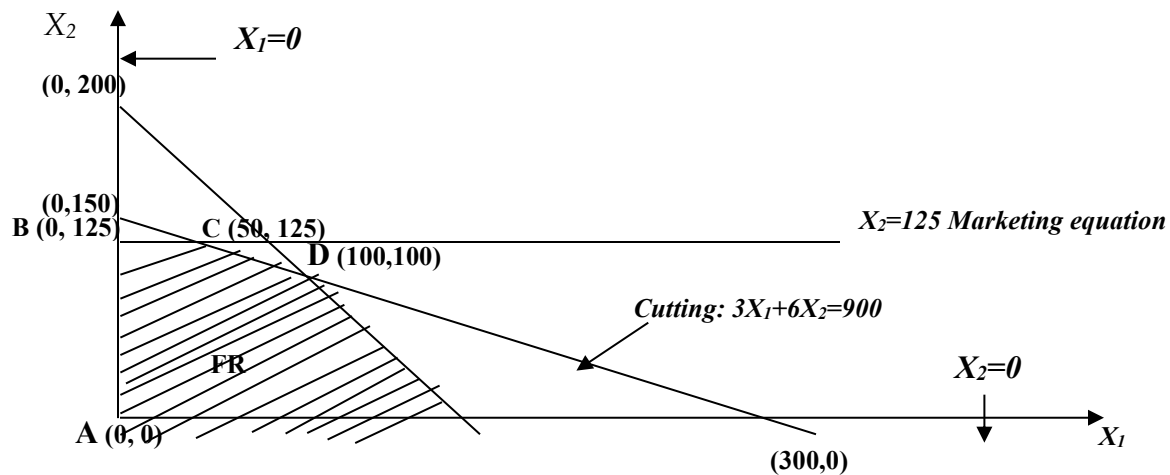
St :

$$3X_1 + 6X_2 \leq 900$$

$$X_1 + X_2 \leq 200$$

$$X_2 \leq 125$$

$$X_1, X_2 \geq 0$$



<u>Corners</u>	<u>Coordinates</u>	<u>MaxZ=8 X₁ + 16X₂</u>
A	(0, 0)	0
B	(0, 125)	2000
C	(50, 125)	2400
D	(100, 100)	2400
E	(200, 0)	1600

Interpretation:

Both **C** and **D** are optimal solutions. Any point on the line segment **CD** will also lead to the same optimal solution.

==>Multiple optimal solutions provide more choices for management to reach their objectives.

2.9.3. Infeasible Solution

A solution is called feasible if it satisfies all the constraints and non-negativity condition. However, it is sometimes possible that the constraints may be inconsistent so that there is no feasible solution to the problem. Such a situation is called ***infeasibility***.

Example:

$$\text{Max } Z = 20X_1 + 30X_2$$

St:

$$2X_1 + X_2 \leq 40$$

$$4X_1 + X_2 \leq 60$$

$$X_1 \geq 30$$

$$X_1, X_2 \geq 0$$

- In the above graph, there is no common point in the shaded area.
- All constraints cannot be satisfied simultaneously and there is no feasible solution to the problem.

Example:

The company has made a contract with an automobile manufacturer to provide 24000 gasoline of *super unleaded*, 42000 gallons of *regular unleaded* and 36000 gallons of *leaded*. The automobile manufacturer wants delivery in not more than 14 days.

Required:

- 41

Solution:

Production per day (in gallons)

Grade of gasoline	A	B	Contract with an automobile manufacturer
-------------------	---	---	--

Super Unleaded	4000	1000	24,000
Regular Unleaded	2000	3000	42,000
Leaded	1000	4000	36,000
Running cost per day	<u>\$1,500</u>	<u>\$2,400</u>	

❖ The automobile manufacturer wants delivery in not more than 14 days.

Let X_1 = The N° of days refinery A should work.

X_2 = The N° of days refinery B should work.

a. LPM of the problem

$$\text{Min } Z = 1500X_1 + 2400X_2$$

St:

$$4000X_1 + 1000X_2 \geq 24000$$

$$2000X_1 + 3000X_2 \geq 42000$$

$$1000X_1 + 2000X_2 \geq 36000$$

$$X_1 \leq 14$$

$$X_2 \leq 14$$

$$X_1, X_2 \geq 0$$

==> To simplify the problem divide by 1000 the constraints

$$\text{Min } Z = 1500X_1 + 2400X_2$$

St:

$$4X_1 + 1X_2 \geq 24$$

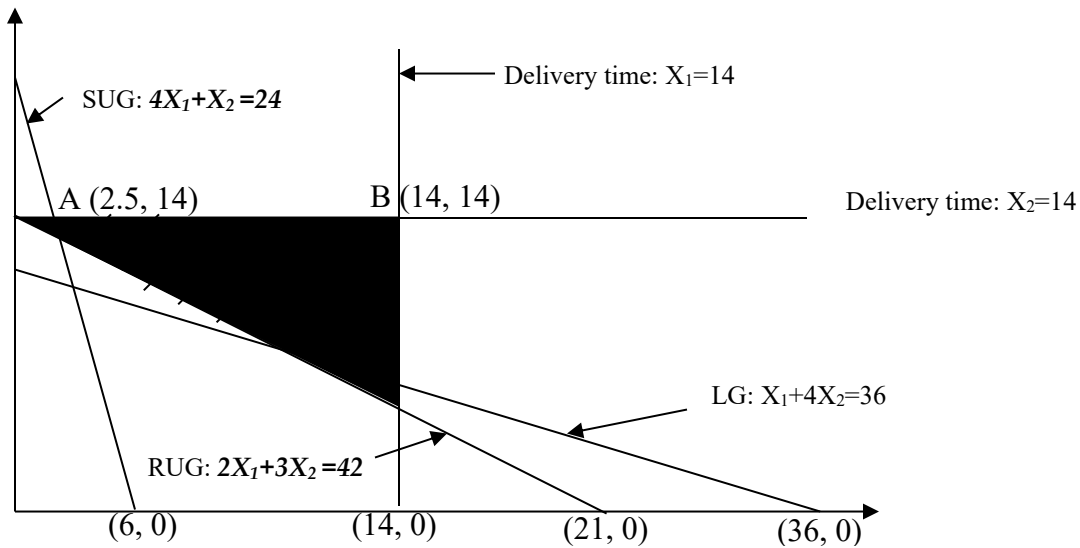
$$2X_1 + 3X_2 \geq 42$$

$$X_1 + 4X_2 \geq 36$$

$$X_1 \leq 14$$

$$X_2 \leq 14$$

$$X_1, X_2 \geq 0$$



Note: Point A, B, C, and D are solved by elimination-substitution method

<u>Corners</u>	<u>Coordinates</u>	<u>MaxZ=1500X₁ + 2400X₂</u>
A	(2.5, 14)	\$37350
B	(14, 14)	54600
C	(14, 5.5)	34200
D	(12, 6)	32400
E	(3, 12)	33300

Interpretation:

The oil company should operate refinery A for 12 days and refinery B for 6 days at a minimum operating cost of \$32,400.

c. Is there any over production?

$$\text{SUG: } 4000X_1 + 1000X_2 \geq 24000$$

$$4000(12) + 1000(6) \geq 24000$$

$$\underline{54000} > \underline{24000}$$

Therefore, 30,000 gallons over production

$$\text{RUG: } 2000X_1 + 3000X_2 \geq 42000$$

$$2000(12) + 3000(6) \geq 42000$$

$$\underline{42000} \geq \underline{42000}$$

Therefore, there is no over production of RUG

$$\text{LG: } 1000X_1 + 4000X_2 \geq 36000$$

$$1000(12) + 1000(6) \geq 36000$$

$$\underline{36000} > \underline{36000}$$

Therefore, no overproduction of LG

2.9.5. Unbounded Solutions

A solution is unbounded if the objective function can be improved without limit without violating the feasibility condition. Here, the objective function value can also be increased infinitely. The solution is unbounded if there are no positive ratios in determining the leaving variable. A negative ratio means that increasing a basic variable would increase resources! A zero ratio means that increasing a basic variable would not use any resources. This condition generally arises because the problem is incorrectly formulated. For example, if the objective function is stated as maximization when it should be a minimization, if a constraint is stated $\geq$ when it should be $\leq$, or vice versa. However, an unbounded feasible region may yield some definite value of the objective function.

Example:

Use the graphical method to solve the following LPP.

1. Max. $Z=3X_1+4X_2$

St: $X_1-X_2 \leq -1 \implies -X_1+X_2 \geq 1$ since the quantity solution is positive

$-X_1+X_2 \leq 0$

$X_1, X_2 \geq 0$

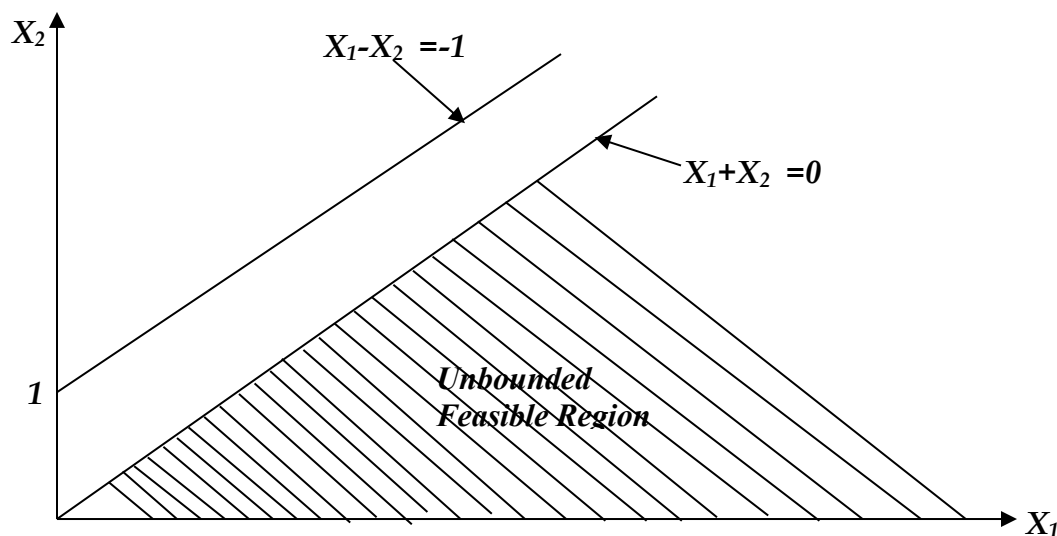


Fig: Unbounded Solution

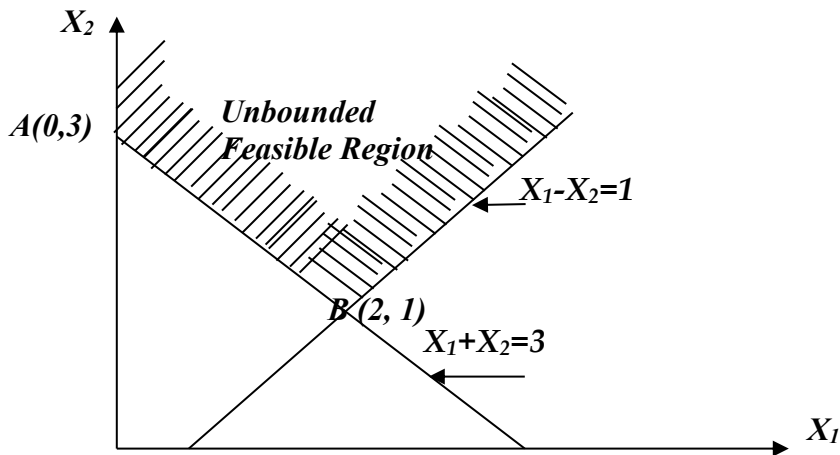
2. Max. $Z=3X_1+2X_2$

St:

$X_1-X_2 \leq 1$

$X_1+X_2 \leq 3$

$X_1, X_2 \geq 0$



Note here that the two corners of the region are $A(0,3)$ and $B(2,1)$. The value of $\text{Max } Z(A) = 6$ and $\text{Max } Z(B) = 8$. But there exist number of points in the shaded region for which the value of the objective function is more than 8. For example, the point $(10, 12)$ lies in the region and the function value at this point is 70 which is more than 8.

Remark:

An unbounded solution does not mean that there is no solution to the given LPP, but implies that there exists an infinite number of solutions.

Exercise:

Use graphical method to solve the following LPP.

1. Max. $Z = 7/4X_1 + 3/2X_2$

St:

$$8X_1 + 5X_2 \leq 320$$

$$4X_1 + 5X_2 \leq 20$$

$$X_1 \geq 15$$

$$X_2 \geq 10$$

$$X_1, X_2 \geq 0$$

Answer: No feasible solution

2. Max. $Z = 3X_1 + 2X_2$

St:

$$-2X_1 + 3X_2 \leq 9$$

$$X_1 - 5X_2 \geq -20$$

$$X_1, X_2 \geq 0$$

Answer: Unbounded solution

3. Max. $Z = 3X_1 + 2X_2$

St:

$$X_1 - X_2 \leq 1$$

$$-3X_1 + X_2 \geq 3$$

$$X_1, X_2 \geq 2$$

Answer: Unbounded solution

4. Max. $Z = X_1 + X_2$

St:

$$X_1 + X_2 \leq 1$$

$$X_1 + X_2 \geq 3$$

$$X_1, X_2 \geq 0$$

Answer: Unbounded solution

5. Max. $Z = 6X_1 - 4X_2$

St:

6. Max. $Z = X_1 + 1/2X_2$

St:

$$2X_1 + 4X_2 \leq 4$$

$$3X_1 + 3X_2 \leq 12$$

$$4X_1 + 8X_2 \geq 16$$

$$5X_1 \leq 10$$

$$X_1, X_2 \geq 0$$

$$X_1 + X_2 \geq 8$$

Answer: Infeasible solution

$$-X_1 + X_2 \geq 4$$

$$X_1, X_2 \geq 0$$

Answer: Infeasible solution

Exercise

I. Solve the following LP problems using the graphical method.

1. Max. $Z = 15X_1 - 10X_2$

2. Max. $Z = 2X_1 + X_2$

St:

St:

$$4X_1 + 6X_2 \leq 360$$

$$X_1 + 2X_2 \leq 10$$

$$3X_1 + 0X_2 \leq 180$$

$$X_1 + X_2 \leq 6$$

$$0X_1 + 5X_2 \leq 280$$

$$X_1 - X_2 \leq 2$$

$$X_1, X_2 \geq 0$$

$$X_1 - 2X_2 \leq 1$$

Answer: $X_1 = 60, X_2 = 20$

$X_1, X_2 \geq 0$

and

Max. $Z = 1,100$

Answer: $X_1 = 4, X_2 = 2$

and Max. $Z = 10$

3. Max. $Z = 10X_1 + 15X_2$

4. Min. $Z = 3X_1 + 2X_2$

St:

St:

$$2X_1 + X_2 \leq 26$$

$$5X_1 + X_2 \geq 10$$

$$2X_1 + 4X_2 \leq 56$$

$$X_1 + X_2 \geq 6$$

$$-X_1 + X_2 \leq 5$$

$$X_1 + 4X_2 \geq 12$$

$$X_1, X_2 \geq 0$$

$$X_1, X_2 \geq 0$$

Answer: $X_1 = 4, X_2 = 2$

Answer: $X_1 = 1, X_2 = 5$

and

Max. $Z = 230$

and Min. $Z = 13$

4. Min. $Z = -X_1 + 2X_2$

4. Min. $Z = 20X_1 + 10X_2$

St:

St:

$$-X_1 + 3X_2 \leq 26$$

$$X_1 + 2X_2 \leq 40$$

$$X_1 + X_2 \leq 6$$

$$3X_1 + 4X_2 \geq 30$$

$$X_1 - X_2 \leq 2$$

$$4X_1 + 3X_2 \geq 60$$

$$X_1, X_2 \geq 0$$

$$X_1, X_2 \geq 0$$

Answer: $X_1 = 2, X_2 = 0$

Answer: $X_1 = 6, X_2 = 12$

and Min. $Z = -2$

and Min. $Z = 240$

II. A manufacturer produces two different models; **X** and **Y**, of the same product. The raw materials **r**₁ and **r**₂ are required for production. At least 18 Kg of **r**₁ and 12 Kg of **r**₂ must be used daily. Almost at most 34 hours of labor are to be utilized. 2Kg of **r**₁ are needed for each model **X** and 1Kg of **r**₁ for each model **Y**. For each model of **X** and **Y**, 1Kg of **r**₂ is required. It takes 3 hours to manufacture a model **X** and 2 hours to manufacture a model **Y**. The profit

realized is \$50 per unit from model **X** and \$30 per unit from model **Y**. How many units of each model should be produced to maximize the profit?

Answer: 10 units of model X, 2 units of model Y and the maximum profit is \$ 560.

III.A manufacturing firm produces two machine parts P_1 and P_2 using milling and grinding machines .The different machining times required for each part, the machining times available on different machines and the profit on each machine part are as given below:

Machine	Manufacturing time required (min)		Maximum time available per week (min)
	P_1	P_2	
Lathe	10	5	25,000
Milling Machine	4	10	2000
Grinding Machine	1	1.5	450
Profit per unit (\$)	<u>\$50</u>	<u>\$100</u>	

Determine the number of pieces of P_1 and P_2 to be manufactured per week to maximize profit.

Answer: $X_1=187.5$, $X_2=125$ and $Max.Z=21,875$

IV.A person requires 10, 12 and 12 units of chemicals **A**, **B** and **C** respectively for his garden. A liquid product contains 5, 2 and 1 units of **A**, **B** and **C** respectively per jar. A dry product contains 1, 2 and 4 units of **A**, **B** and **C** per carton. If the liquid product sells for \$3 per jar and the dry product sells \$2 per carton, how many of each should be purchased in order to minimize cost and meet the requirement?

Answer: 1 Unit of liquid, 5 units of dry product and $Min.Z=\$8$

6. Degeneracy

In the process of developing the next simplex tableau for a tableau that is not optimal, the leaving variable must be identified. This is normally done by computing the ratios of values in the quantity column and the corresponding row values in the entering variable column, and selecting the variable whose row has the smallest non-negative ratio. Such an occurrence is referred to degeneracy, because it is theoretically possible for subsequent solutions to cycle (i.e., to return to previous solutions). There are ways of dealing with ties in a specific fashion; however, it will usually suffice to simply select one row (variable) arbitrarily and proceed with the computations.

2.10. Limitations of linear programming

1. In linear programming uncertainty is not allowed, i.e., LP methods are applicable only when values for costs, constraints, etc. are known, but in real life such factors may be unknown.
2. According to the LP problem, the solution variables can have any value, whereas sometimes it happens that some of the variables can have only integer values. For example, in finding how many machines to be produced; only integer values of decision variables are meaningful. Except when the variables have large values, rounding the solution to the nearest integer will not yield an optimal solution. Such situations justify the use of Integer Programming.
3. Many times, it is not possible to express both the objective function and constraints in linear form.

2.11. Sensitivity (Post-Optimality) Analysis

It begins with the **final simplex tableau**. Its purpose is to explore how changes in any of the parameters of a problem, such as the coefficients of the constraints, coefficients of the objective function, or the right-hand side values, would affect the solution. This kind of analysis is beneficial to a decision maker in a number of ways.

In preceding parts, the formulation and solution of LPPs treated the parameters of a problem as if they were fixed and known. The goal of analysis is to determine the optimal value of the decision variables in the context. Unfortunately, in practice, the parameters of a problem often are no more than educated guesses. Consequently, the optimal solution may or may not be optimal, depending on how sensitive that solution is to alternate values of parameters. Hence, a decision maker usually would want to perform sensitivity analysis before implementing a solution. If the additional analysis satisfies the decision maker that the solution is reasonable, s/he can proceed with greater confidence in using the solution. Conversely, if sensitivity analysis indicates that the solution is sensitive to changes that are within the realm of possibility, the decision maker will be warned that more precise information on parameters is needed.

Example: Max. $Z = 60X_1 + 50X_2$

$$\text{S.t: } 4X_1 + 10X_2 \leq 100$$

$$2X_1 + X_2 \leq 22$$

$$3X_1 + 3X_2 \leq 39$$

$$X_1, X_2 \geq 1$$

2.11.1. Change in the RHSV of a Constraint

The first step in determining how a change in the RHSV of a constraint (e.g., the amount of scarce resource that is available for use) would influence the optimal solution is to examine the *shadow prices* in the final tableau. These are the values in the Z row in the slack columns. The final tableau for the microcomputer problem is shown below.

Table 2.11.1 Shadow prices in the microcomputer final tableau

C _j	60	50	0	0	0	
Basis	X ₁	X ₂	S ₁	S ₂	S ₃	Quantity
S ₁ 0	0	0	1	6	-16/3	24
X ₁ 60	1	0	0	1	-1/3	9
X ₂ 50	0	1	0	-1	2/3	4
Z	60	50	0	10	40/3	740
C _j - Z _j	0	0	0	-10	-40/3	

Negatives of shadow prices

The shadow prices are 0, 10, and 40/3. Note that the negatives of the shadow prices appear in the bottom row of the table. Because they differ only in sign, the values in the bottom row are sometimes also referred to as shadow prices.

A shadow price is marginal value; it indicates the impact that a one-unit change in the amount of a constraint would have on the value of the objective function. More specifically, a shadow price reveals the amount by which the value of the objective function would increase if the level of the constraint was increased by one unit. The above table reveals that the shadow prices are \$0 for assembly time, \$10 for inspection time, and \$40/3 for storage space. This tells us that if the amount of assembly time was increased by one hour, there would be no effect on profit; if inspection time was increased by one hour, the effect would be to increase profit by \$10; and if storage space was increased by one cubic foot, profit would increase by \$40/3, or \$13.33. The reverse also holds. That is, by using the negative sign of each shadow price, we can determine how much a one-unit decrease in the amount of

each constraint would decrease profit. Hence, a one-unit decrease in assembly time would have no effect because its shadow price is \$0. However, a one-unit decrease in inspection time would decrease profit by \$10; and a one-unit decrease in storage space would decrease profit \$13.33. In effect, shadow prices are a manager's "window" for gauging the impact that changes have on the value of the objective function.

What the shadow prices do not tell us is the extent to which the level of a scarce resource can be changed and still have the same impact per unit. For example, we now know that a one unit increase in inspection time will increase profit by \$10. Because of linearity, a two unit increase in inspection time will mean an increase of \$20. However, at some point, the ability to use additional amounts of a resource will disappear because of the fixed amounts to the other constraints. In other words, limits on assembly time and storage space mean that only a certain amount of additional inspection time can be used and still have a feasible solution.

Conversely, a similar situation can arise when considering the potential impact of a decrease in the RHS of a constraint. For instance, the assembly constraint has a shadow price of \$0, which indicates that a one-unit decrease in the amount of assembly time available will have no impact on profit. However, we know that some amount of assembly time is required to produce the computers, which mean that it cannot be decreased indefinitely. At some point, fewer assembly hours will restrict output. Therefore, we need to determine the **range** over which we can change the RHSV and still have the same shadow price. This is called the **range of feasibility**, or the right hand side range.

Table 2.11.2 Determining the Range of Feasibility of the Microcomputer Problem

Assembly	Inspection	Storage	Quantity
0 S ₁	0 S ₂	0 S ₃	
1	6	-16/3	24
0	1	-1/3	9
0	-1	2/3	4
0	10	40/3	740
0	-10	-40/3	

	Assembly	Inspection	Storage
	$24/1 = +24$	$24/6 = +4$	$\frac{24}{-16/3} = -4.5$
	$9/0 = \text{Undefined}$	$9/1 = +9$	$\frac{9}{-1/3} = -27$
	$4/0 = \text{Undefined}$	$4/-1 = -4$	$\frac{4}{2/3} = +6$
Original amount	100 hours	22 hours	39 cubic feet
Upper limit	None	$22+4 = 26$	$39+4.5 = 43.5$
Lower limit	$100-24 = 76$	$22-4 = 18$	$39-6 = 33$
Range	$76-\infty$	$18-26$	$33-43.5$

Unlike the simplex calculations, negative ratios are acceptable. Two of these ratios indicate the extent to which the storage space constraint level can be changed and still have the current shadow price remain valid: The smallest positive ratio indicates how much the constraint level can be decreased before it reaches the lower limit of its range of feasibility. Thus, the storage constraint can be *decreased* by 6 cubic feet before it hits the lower of its range of feasibility. Conversely, the smallest negative ratio (i.e., the negative ratio closest to 0) indicates how much the storage constraint can be *increased* before it reaches its upper limit of feasibility. Hence, the storage level can be increased by 4.5 cubic feet before it hits that upper limit.

Perhaps it seems strange that a positive ratio relates to a decrease and a negative ratio to an increase. However, the positive ratio indicates the amount that must be added to the lower limit to achieve the current constraint level (i.e., how much the current level is above the lower limit). Thus, knowing the current level, we must subtract that amount to obtain the lower limit. Thus, for the storage constraint, the current level is 39. thus, we have:

$$\text{Lower limit} + 6 = 39, \text{ so } \text{Lower limit} = 39-6.$$

Similarly, the smallest negative ratio reveals how much the current level is below the upper limit, so that amount must be added to the current level to determine the upper limit. For the storage constraint, that is:

Upper limit - 4.5 = 39, so Upper limit = 39+4.5

The same general rule always applies when computing the upper and lower limits on the range of feasibility for the maximization problem:

Allowable decrease: The smallest positive ratio.

Allowable increase: The negative ratio closest to zero.

For a minimization problem, the rules are reversed: The allowable decrease is the negative ratio closest to zero and the allowable increase is the smallest positive ratio.

2.11.2. Change in an Objective Function Coefficient

A manager or other decision maker often finds it useful to know how much the contribution of a given decision variable to the objective function can change without changing the optimal solution. Such changes may occur because of changed costs, new pricing policies, product or process design changes, as well as other factors. In some instances, the changes reflect improved information.

There are two cases to consider: changes for a variable that is not currently in the solution mix, and changes for a variable that is currently in the solution mix. The range over which a non-basic variable's objective function coefficient can change without causing that variable to enter the solution mix is called its *range of insignificance*. The range over which the objective function coefficient of a variable that is in solution can change without changing the optimal values of the decision variables is called its *range of optimality*. Note, however, that such a change would change the optimal value of the objective function.

Example: Given the final tableau in Table 2.9.3, determine the range over which the objective function coefficient of variable X_3 could change without changing optimal solution.

Table 2.11.3 Final Simplex Tableau

Cj	120	105	112	0	0	0	
Basis	X_1	X_2	X_3	S_1	S_2	S_3	Quantity
X_1 120	0	0	0.06	0.14	-0.08	0	16.8
X_2 105	1	1	1.10	-0.10	0.20	0	8
S_3 0	0	0	0.88	-0.28	0.16	1	16.4
Zj	0	0	122.7	6.30	11.40	0	2,856
Cj-Zj	0	0	-10.7	-6.30	-11.40	0	

Solution: Because X_3 is not in solution, its objective function coefficient would need to increase in order for it to come into solution. The amount of increase must be greater than its $C_j - Z_j$ value, which is -10.7. Therefore, its objective coefficient function must increase by more than \$10.70. Because its current value is \$112, as indicated at the top of Table 2.9.3, the current solution will remain optimal as long as its objective function coefficient does not exceed \$112+\$10.70. Since an increase would be needed to bring it into solution, any value of its objective function coefficient would keep it out of solution. Hence, the range insignificance for X_3 is \$112.70 or less.

For variables which are in solution, the determination of the range of optimality requires a different approach. Hence, the rules (for both maximization and minimization problems) are:

Allowable increase: The smallest positive ratio of $C_j - Z_j$ value and the variable substitution rate.

Allowable decrease: The smallest negative ratio of $C_j - Z_j$ value and the variable substitution rate.

(Note: If there is no positive ratio, there is no upper limit on that variable's objective function coefficient.)

C _j	60	50	0	0	0	
Basis	X ₁	X ₂	S ₁	S ₂	S ₃	Quantity
S ₁ 0	0	0	1	6	-16/3	24
X ₁ 60	1	0	0	1	-1/3	9
X ₂ 50	0	1	0	-1	2/3	4
Z _j	60	50	0	10	40/3	740
C _j -Z _j	0	0	0	-10	-40/3	

For X₁ we find:

Column X₁ X₂ S₁ S₂ S₃

C_j-Z_j value 0/1 = 0 0/0 = Undefined 0/0 = Undefined -10/1 = -10 -40/3 = +40

X₁ value -1/3

The smallest positive ratio is +40. Therefore, the coefficient of X₁ can be increased by \$40 without changing the optimal solution. The upper end of its range of optimality is this amount added to its current (original) value. Thus, its upper end is \$60+\$40 = \$100. Also, the smallest negative ratio is -10; therefore, the X₁ coefficient can be decreased by as much as \$10 from its current value, making the lower end of the range equal to \$60-\$10 = \$50.

For X₂ we find:

Column X₁ X₂S₁ S₂ S₃

C_j - Z_j value 0/0 = Undefined 0/1 = 0 0/0 = Undefined -10/-1 = +10 -40/3 = -20

X₂ value 2/3

The smallest positive ratio is +10. This tells us that the X₂ coefficient in the objective function could be increased by \$10 to \$50+\$10 = \$60. The smallest positive ratio is -20, which tells us the X₂ coefficient could be decreased by \$20 to \$50-\$20 = \$30. Hence, the range of optimality for the objective function coefficient of X₂ is \$30 to \$60.

2.12. Duality in Linear Programming Problem

Every LPP has another LPP associated with it, which is called its dual. The first way of stating a linear problem is called the Primal of the problem. The second way of stating the same problem is called the dual. The optimal solutions for the primal and the dual are equivalent, but they are derived through alternative procedures. The dual is kind of a "mirror image".

Table 2.12.1 Primal-Dual Relationship

Primal	Dual
Objective is minimization	Objective is maximization and vice versa
$\geq$ Type constraints	$\leq$ Type constraints
Number of columns	Number of rows
Number of rows	Number of columns
Number of decision variables	Number of constraints
Number of constraints	Number of decision variables
Coefficient of objective function	RHS value
RHS value	Coefficient of objective function

Duality Advantage

1. The dual form provides an alternative form.
2. The dual reduces the computational difficulties associated with some formulation.
3. The dual provides an important economic interpretation concerning the value of scarce resources used.

Example 1: Max. Z: $60X_1 + 50X_2$ — RHSV

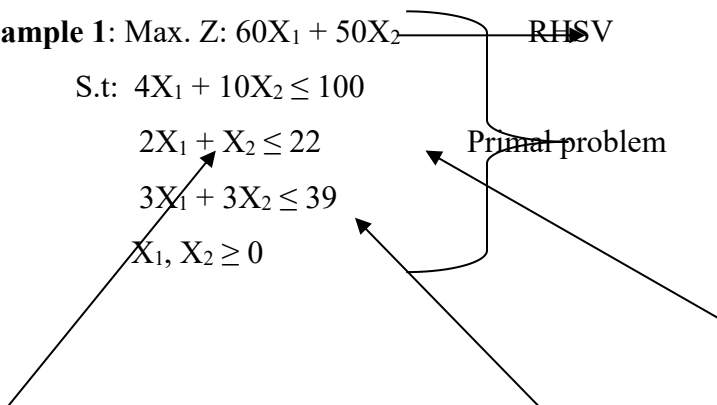
S.t: $4X_1 + 10X_2 \leq 100$

$2X_1 + X_2 \leq 22$

$3X_1 + 3X_2 \leq 39$

$X_1, X_2 \geq 0$

Primal problem



Objective function coefficient of the dual

1st Constraint of the dual

2nd Constraint of the dual

Required:

1. Formulate the dual problem.
2. Find the primal solution.
3. Find the dual problem.

Solution: 1) Dual problem

$$\text{Min. } Z: 100Y_1 + 22Y_2 + 39Y_3$$

$$\text{S.t: } 4Y_1 + 2Y_2 + 3Y_3 \geq 60$$

$$10Y_1 + Y_2 + 3Y_3 \geq 50$$

$$Y_1, Y_2, Y_3 \geq 0$$

2)

C _j		60	50	0	0	0	
Basis		X ₁	X ₂	S ₁	S ₂	S ₃	RHSV
S ₁	0	0	0	1	6	-16/3	24
X ₁	60	1	0	0	1	-1/3	9
X ₂	50	0	1	0	-1	2/3	4
Z		60	50	0	10	40/3	740
C _j - Z _j		0	0	0	-10	-40/3	

X₁ = 9, X₂ = 4, S₁ = 24, Z Max. = **Birr 740.**

3) **Primal** **Dual**

S ₁	Y ₁ = 0
S ₂	Y ₂ = 10
S ₃	Y ₃ = 40/3
X ₁	S ₁ = 0
X ₂	S ₂ = 0

2.13. Correspondence Between Primal and Dual Optimal Solutions

1) Values for the non-basic variables of the primal are given by the base row of the dual solution, under the slack variables (if there are any), neglecting the negative sign if any and under the artificial variables (if there is no slack variable in a constraint) neglecting the negative sign if any, after deleting the constraint M.

- 2) Values for the slack variables of the primal are given by the row under the non-basic variables of the dual solution neglecting the negative sign if any.
- 3) The value of the objective function is same for primal and dual solutions.

Example 2: Max. Z: $50X_1 + 40X_2$

S.t: $3X_1 + 5X_2 \leq 150$ Assembly time

$X_2 \leq 20$ Portable display

$8X_1 + 5X_2 \leq 300$ Warehouse space

$X_1, X_2 \geq 0$

Method (I) Using the Primal Problem

The final simplex tableau is shown here:

C _j		50	40	0	0	0	
Basis		X ₁	X ₂	S ₁	S ₂	S ₃	RHSV
X ₂	40	0	1	8/25	0	-3/25	12
S ₂	0	0	0	-8/25	1	3/25	8
X ₁	50	1	0	-5/25	0	5/25	30
Z _j		50	40	14/5	0	26/5	1980
C _j - Z _j		0	0	-14/5	0	-26/5	

The optimal solution to the primal problem is:

$X_1 = 30, X_2 = 12, S_1 = 0, S_2 = 8, S_3 = 0$.

The optimal value of the objective function is **Birr 1980.**

Primal

S₁

S₂

S₃

X₁

X₂

Dual

$Y_1 = 14/5$

$Y_2 = 0$

$Y_3 = 26/5$

$S_1 = 0$

$S_2 = 0$

Method (II) Using the Dual Problem

Min. Z: $150Y_1 + 20Y_2 + 300Y_3$

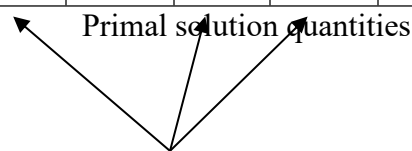
S.t: $3Y_1 + 0Y_2 + 8Y_3 \geq 50$

$5Y_1 + Y_2 + 5Y_3 \geq 40$

$Y_1, Y_2, Y_3 \geq 0$

The final simplex tableau is shown here:

Cj		150	20	0	0	0	M	M	
Basis		Y ₁	Y ₂	Y ₃	S ₁	S ₂	A ₁	A ₂	RHSV
Y ₃	300	0	-3/25	1	-1/5	3/25	1/5	-1/25	26/5
Y ₁	150	1	8/25	0	1/5	-8/25	-1/5	8/25	14/5
Z _j		150	12	300	-30	-12	30	12	1980
Cj - Zj		0	8	0	30	12	M-30	M-12	



Because all the entries in the net evaluation row are greater than or equal to zero, the optimal solution has been reached;

It is $Y_1 = 14/5$, $Y_2 = 0$, $Y_3 = 26/5$, $S_1 = 0$, and $S_2 = 0$.

The optimal value of the objective function is **Birr 1980**.

$$S_1 = A_1 = X_1 = 30$$

$$S_2 = A_2 = X_2 = 12$$

$$S_3 = S_1 = 0$$

$$Y_1 = S_1 = 0$$

$$Y_2 = S_2 = 8$$

$$Y_3 = S_3 = 0$$

Example 3: Formulate the dual when the primal has mixed constraints:

$$\text{Max. Z: } 50X_1 + 80X_2$$

$$\text{S.t: } 3X_1 + 5X_2 \leq 45$$

$$(4X_1 + 2X_2 \geq 16) \quad \dots (-1)$$

$$6X_1 + 6X_2 = 30 \quad \text{We need two constraints}$$

$$X_1, X_2 \geq 0 \quad 6X_1 + 6X_2 \leq 30$$

$$(6X_1 + 6X_2 \geq 30) \quad \dots (-1)$$

$$-6X_1 - 6X_2 \leq -30$$

We need to rearrange them,

$$\text{Max. Z: } 50X_1 + 80X_2$$

$$\text{S.t: } 3X_1 + 5X_2 \leq 45$$

$$-4X_1 - 2X_2 \leq -16$$

$$6X_1 + 6X_2 \leq 30$$

$$-6X_1 - 6X_2 \leq -30$$

$$X_1, X_2 \geq 0$$

Dual Problem

$$\text{Min. } Z: 45Y_1 - 16Y_2 + 30Y_3 - 30Y_4$$

$$\text{S.t: } 3Y_1 - 4Y_2 + 6Y_3 - 6Y_4 \geq 50$$

$$5Y_1 - 2Y_2 + 6Y_3 - 6Y_4 \geq 80$$

$$Y_1, Y_2, Y_3, Y_4 \geq 0$$

2.14. Economic Interpretation of the Dual Variables

Dual variables have an interpretation from the cost or economic point of view.

$$\text{Min. } Z = 600X_1 + 400X_2$$

$$\text{S. t: } 1500X_1 + 1500X_2 \geq 20,000$$

$$3000X_1 + 1000X_2 \geq 40,000$$

$$2000X_1 + 5000X_2 \geq 44,000$$

$$X_1, X_2 \geq 0$$

The dual of this primal is:

$$\text{Max. } Z = 20,000Y_1 + 40,000Y_2 + 44,000Y_3 \quad \dots 2.1$$

$$\text{S. t: } 1500Y_1 + 3000Y_2 + 2000Y_3 \leq 600 \quad \dots 2.2$$

$$1500Y_1 + 1000Y_2 + 5000Y_3 \leq 400 \quad \dots 2.3$$

$$Y_1, Y_2, Y_3 \geq 0 \quad \dots 2.4$$

Optimal solution to the primal has been obtained:

$$X_1 = 12 \text{ days,}$$

$$X_2 = 4 \text{ days,}$$

$$\text{MIN. } Z = 8,800 \text{ monetary units}$$

Optima solution to the dual is found to be:

$$Y_1 = 0,$$

$$Y_2 = 11/65,$$

$$Y_3 = 3/65$$

Max. $Z = 8,800$ monetary units

Now, as the RHS of constraints (2.2 and 2.3), denotes the monetary units, the left hand side must also be expressed in monetary units. In the first term $1500Y_1$ of the first constraint, 1500 is the number of bottles of whisky produced per day. Hence, Y_1 must denote the cost of producing one bottle of whisky. Similarly, Y_2 denotes the cost of producing a bottle of beer and Y_3 denotes the cost of producing a bottle of fruit juices. Y_1 , Y_2 and Y_3 are called the shadow prices of whisky, beer and fruit juices respectively. They represent not the actual market prices but the true accounting values or the imputed values of the three drinks.

The objective function of the dual is to maximize the total accounting values of the drinks produced per month. The two constraints of the dual ensure that the total accounting value of the daily output of each plant must remain less than the daily cost operating the plant. The shadow prices represent the values that the company should set on its resources in order to reflect their value to society, while the constraints ensure the internal price can't be set to get more value from a drink than what the company puts into it. It means that in a situation of equilibrium, the laws of economics for society do not require any profit.

The values of $Y_1 = 0$, $Y_2 = 11/65$ and $Y_3 = 3/65$ represent the shadow prices of whisky, beer and fruit juices respectively. $Y_1 = 0$ means that the accounting value of whisky is zero and that it is produced in surplus as a by-product.

■ SUMMARY

Linear Programming (LP) is a mathematical modeling technique useful for economic allocation of “scarce” or “limited” resources, such as labour, material, machine, time, warehouse, space, capital, etc. to several competing activities such as products, services, jobs, new equipment, projects etc, on the basis of a given criterion of optimality. Linear programming models are used to find optimal solutions to constrained optimization problems. In order for linear programming models to be used, the problems must involve a single

objective, a linear objective function and linear constraints and have known and constant numerical values. Linear programming models are composed of decision variables and numerical values that are arranged into an objective function and a set of constraints. The constraints are restrictions that can pertain to any decision variable or to combination of decision variables. In general, variables are not allowed to have negative values. These restrictions are referred to as non-negativity constraints. Linear programming models are widely used. Among the application of these models are problems that involve product mix, portfolio selection, distribution, assignments and production and inventory planning. Graphic method has a limited application in solving linear programming problem which have two variables. To put simply, the graphical technique can only handle problems involving two variables. But if the linear programming problem has more than two variables, the method either fails or becomes too complex to solve the problem. Simplex method, on the other hand, is the comprehensive method which can be applied in solving linear programming problems having any number of variables.

Self Test Exercise 2

Part I: Write “True” if the statement is correct and “False” if the statement is incorrect.

1. All linear programming problems may be solved only graphic method.
2. A two variable linear programming cannot be solved by simplex problem.
3. Sensitivity analysis answers ‘what if’ questions to help the decision maker.

Part II: Choose the correct answer & encircle the letter of your choice.

1. Assumptions of liner programming include:

A. Linearity	D. Divisibility
B. Additivity	E. All of the above
C. Certainty	
2. Divisibility assumption in linear programming implies:

A. Resource can be divided among producers
B. Products can be divided among customers

- C. Decision variables may not take on integral value
- D. Decision variables may take fractional value

Part III: Provide brief explanation for each of the following questions.

1. *What does it mean linear programming?*
2. *What are the components of linear programming?*
3. *What are the assumptions of linear programming?*
4. *Solve the following linear programming problem using both Graphic and simplex method: Minimize Z: $x_1 + x_2$*

Subject to the constraint:

$$2x_1 + 4x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$

$$x_1, x_2 \geq 0$$

UNIT THREE: TRANSPORTATION PROBLEMS AND ASSIGNMENT PROBLEMS

3. Introduction

Dear learners, in the preceding units, you learned the meaning, applications and methods of solving the linear programming problem. In this unit you will learn two special classes of linear programming known as the Transportation Problem and Assignment problem. The transportation problem typically deal with a special class of linear programming problem in which the objective is to “transport” or distribute a single commodity from several „sources” (also called origins or supplies or capacity centers) to different „destinations” (also such as demands or requirement (enters) at a minimum total cost. Assignment problems are special class of linear programming problems which involve determining most efficient assignment of people to Projects, jobs to machines, salespersons to territories, contracts to bidders, classes to rooms and so on in one-to-one basis. The desired objective is most often to minimize total costs or time required to perform the tasks at hand. The chapter starts through providing the general skeletal form of the transportation model. Then different methods of solving the transportation problem will be described. Some of the special cases like unbalanced transportation problem, degeneracy, profit maximization will also be discussed. Concepts and methods of solving assignment problems will be discussed at the end of the unit.

Learning Objectives: The major aim of this unit is to introduce you with the applications and methods of solving transportation and assignment problems.

At the end of this unit, you will be able to:

- Explain different terms which are required to formulate a transportation problem;
- Understand different methods of solving the transportation problem;
- Describe the major areas where the transportation problem is effectively applied;
- Identify the especial cases of transportation problem;
- Clarify the necessary modifications required to solve the especial cases of transportation problem;

- Distinguish the problems that can be solved using assignment method;
- Solve various assignment problems.

3.1. Transportation Problem

Dear learners, Do you have any idea about Transportation problem? Good! Transportation problem is a special class of linear programming problem in which the objective is to “transport” or distribute a single commodity from several „sources” (also called origins or supplies or capacity centers) to different „destinations” (also such as demands or requirement (enters) at a minimum total cost. One important application of linear programming is in the area of physical distribution (transportation) of goods and services from several supply origins to several demand destinations. It is easy to express a transportation problem mathematically in terms of an LP model, which can be solved by the simplex method. But because it involves a large number of variables and constraints, it takes a long time to solve it. However, transportation algorithms, namely the Stepping Stone Method and the MODI (Modified Distribution) Method, have been developed for this purpose.

The transportation problem is one of the subclasses of linear programming problem where the objective is to transport various quantities of a single homogeneous product that are initially stored at various origins, to different destinations in such a way that the total transportation is minimum.

F.I. Hitchcock developed the basic transportation problem in 1941. However it could be solved for optimally as an answer to complex business problem only in 1951, when George B. Dantzig applied the concept of Linear Programming in solving the Transportation models.

Transportation models or problems are primarily concerned with the optimal (best possible) way in which a product produced at different factories or plants (called supply origins) can be transported to a number of warehouses (called demand destinations). The objective in a transportation problem is to fully satisfy the destination requirements within the operating production capacity constraints at the minimum possible cost. Whenever there is a physical movement of goods from the point of manufacture to the final consumers through a variety

of channels of distribution (wholesalers, retailers, distributors etc.), there is a need to minimize the cost of transportation so as to increase the profit on sales. Transportation problems arise in all such cases. It aims at providing assistance to the top management in ascertaining how many units of a particular product should be transported from each supply origin to each demand destinations to that the total prevailing demand for the company's product is satisfied, while at the same time the total transportation costs are minimized.

A scooter production company produces scooters at the units situated at various places (called origins) and supplies them to the places where the depot (called destination) are situated. Here the availability as well as requirements of the various depots are finite and constitute the limited resources. This type of problem is known as distribution or transportation problem in which the key idea is to minimize the cost or the time of transportation.

In previous lessons we have considered a number of specific linear programming problems. Transportation problems are also linear programming problems and can be solved by simplex method but because of practical significance the transportation problems are of special interest and it is tedious to solve them through simplex method. Is there any alternative method to solve such problems?

3.2. Mathematical Formulation of Transportation Problem

Let there be three units, producing scooter, say, A1, A2 and A3 from where the scooters are to be supplied to four depots say B1, B2, B3 and B4. Let the number of scooters produced at A1, A2 and A3 be a_1 , a_2 and a_3 respectively and the demands at the depots be b_1 , b_2 , b_3 and b_4 respectively. We assume the condition $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 + b_4$ i.e., all scooters produced are supplied to the different depots. Let the cost of transportation of one scooter from A1 to B1 be c_{11} . Total numbers of scooters to be transported from A1 to all destinations, i.e., B1, B2, B3, and B4 must be equal to a_1 .

$$x_{11} + x_{12} + x_{13} + x_{14} = a_1 \quad (1)$$

Similarly, from A2 and A3 the scooters transported are equal to a_2 and a_3 respectively.

$$x_{21} + x_{22} + x_{23} + x_{24} = a_2 \quad (2)$$

And
$$x_{31} + x_{32} + x_{33} + x_{34} = a_3 \quad (3)$$

On the other hand it should be kept in mind that the total number of scooters delivered to B1 from all units must be equal to b1, i.e,

$$X_{11} + x_{21} + x_{31} = b_1 \quad (4)$$

Similarly, $x_{12} + x_{22} + x_{32} = b_2 \quad (5)$

$$X_{13} + x_{23} + x_{33} = b_3 \quad (6)$$

$$X_{14} + x_{24} + x_{34} = b_4 \quad (7)$$

With the help of the above information we can construct the following table:

Table 3.1

Depot Unit	To B1	To B	To B3	To B4	Stock
From A1	$x_{11}(c_{11})$	$A_1x_{12}(c_{12})$	$x_{13}(c_{13})$	$x_{14}(c_{14})$	a1
From A2	$x_{21}(c_{21})$	$x_{22}(c_{22})$	$x_{23}(c_{23})$	$x_{24}(c_{24})$	a2
From A3	$x_{31}(c_{31})$	$x_{32}(c_{32})$	$x_{33}(c_{33})$	$x_{34}(c_{34})$	a3
Requirement	b1	b2	b3	b4	

The cost of transportation from A_i ($i=1, 2, 3$) to B_j ($j=1, 2, 3, 4$) will be equal to

$$S = \sum_{i,j} c_{ij} x_{ij},$$

Where the symbol put before $c_{ij} x_{ij}$ signifies that the quantities $c_{ij} x_{ij}$ must be summed over all $i = 1, 2, 3$ and all $j = 1, 2, 3, 4$.

Thus we come across a linear programming problem given by equations (1) to (7) and a linear function (8). We have to find the non-negative solutions of the system such that it minimizes the function (8).

Note: We can think about a transportation problem in a general way if there are m sources (say $A_1, A_2, \dots, A_m$) and n destinations (say $B_1, B_2, \dots, B_n$). We can use a_i to denote the

quantity of goods concentrated at points $A_i (i=1,2,\dots, m)$ and b_j denote the quantity of goods expected at points $B_j (j=1,2,\dots,n)$. We assume the condition.

$a_1+a_2+\dots+a_m=b_1+b_2+\dots+b_n$ implying that the total stock of goods is equal to the summed demand for it.

Some Definitions

The following terms are to be defined with reference to the transportation problems:

(A) Feasible Solution (F.S.) a set of non-negative allocations $x_{ij} \geq 0$ which satisfies the row and column restrictions is known as feasible solution.

(B) Basic Feasible Solution (B.F.S.) a feasible solution to an m -origin and n -destination problem is said to be basic feasible solution if the number of positive allocations are $(m+n-1)$. If the number of allocations in a basic feasible solutions are less than $(m+n-1)$, it is called degenerate basic feasible solution (DBFS) (otherwise non-degenerate).

(C) Optimal Solution

A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

3.3. Solved Examples on Transportation Problem

Let us consider the numerical version of the problem stated in the introduction and the mathematical formulation of the same in the next section, as below in Table 2.

Table3. 2

Depot Unit	B1	B2	B3	B4	Stock
A1	$c_{11}=2$	$c_{12}=3$	$c_{13}=5$	$c_{14}=1$	$a_1=8$
A2	$c_{21}=7$	$c_{22}=3$	$c_{23}=4$	$c_{24}=6$	$a_2=10$
A3	$c_{31}=4$	$c_{32}=1$	$c_{33}=7$	$c_{34}=2$	$a_3=20$
Requirement	$b_1=6$	$b_2=8$	$b_3=9$	$b_4=15$	$=38$

(All terms are in hundreds)

$\sum b_{ji}$

In order to find the solution of this transportation problem we have to follow the steps given below.

(A) Initial basic feasible solution

(B) Test for optimization.

Let us consider these steps one by one.

(A) Initial Basic Feasible Solution: There are three different methods to obtain the initial basic feasible solution viz.

(I) North-West corner rule

(II) Lowest cost entry method

(III) Vogel's approximation method

In the light of above problem let us discuss one by one.

3.3.1. North-West corner rule

In this method we distribute the available units in rows and column in such a way that the sum will remain the same. We have to follow the steps given below.

(a) Start allocations from north-west corner, i.e., from (1, 1) position. Here $\min(a_1, b_1)$, i.e., $\min(8, 6) = 6$ units. Therefore, the maximum possible units that can be allocated to this position is 6, and write it as 6(2) in the (1, 1) position of the table. This completes the allocation in the first column and cross the other positions, i.e., (2, 1) and (3, 1) in the column.

(See Table 3)

Table 3.3

Depot Unit	B1	B2	B3	B4	Stock
A1)	6(2)				8-6=2
A2	×				10
A3	×				20
Requirement	6-6=0	8	9	15	32

(b) After completion of step (a), come across the position (1, 2). Here $\min(8-6, 8)=2$ units can be allocated to this position and write it as 2(3). This completes the allocations in the first row and cross the other positions, i.e., (1, 3) and (1, 4) in this row (see Table 4).

Table3. 4

Depot Unit	B1	B2	B3	B4	Stock
A1	6(2)	2(3)	x	X	2-2=0
A2	x				10
A3	x				20
Requirement	0	8-2=6	9	15	30

(c) Now come to second row, here the position (2, 1) is already been struck off, so consider the position (2, 2). Here $\min(10, 8-2)=6$ units can be allocated to this position and write it as 6(3). This completes the allocations in second column so strike off the position (3, 2) (see Table 5)

Table 3.5

Depot Unit	B1	B2	B3	B4	Stock
A1	6(2)	2(3)	X	X	0
A2	x	6(3)			10-6=4
A3	x	x			20
Requirement	0	0	9	15	24

(d) Again consider the position (2, 3). Here, $\min(10-6, 9)=4$ unit scan be allocated to this position and write it as 4(4). This completes the allocations in second row so struck off the position (2, 4) (see Table 6).

Table 3.6

Depot Unit	B ₁	B ₂	B ₃	B ₄	Stock
A ₁	6(2)	2(3)	X	X	0
A ₂	x	6(3))	4(4	X	0
A ₃	x	x			20
Requirement	0	0	9-4=5	15	20

(e) In the third row, positions (3, 1) and (3, 2) are already been struck off so consider the position (3, 3) and allocate it the maximum possible units, i.e., $\min(20, 9-4) = 5$ units and write it as 5(7). Finally, allocate the remaining units to the position (3,4), i.e., 15 units to this position and write it as 15(2). Keeping in mind all the allocations done in the above method complete the table as follows:

Table 3.7

Depot Unit	B ₁	B ₂	B ₃	B ₄	Stock
A ₁	6(2)	2(3)	X	X	8
A ₂	x	6(3)	4(4)	X	10
A ₃	x	x	5(7)	15(2)	20
Requirement	6	8	9	15	38

From the above table calculate the cost of transportation as $6 \times 2 + 2 \times 3 + 6 \times 3 + 4 \times 4 + 5 \times 7 + 15 \times 2 = 12 + 6 + 18 + 16 + 35 + 30 = 117$ i.e., Rs. 11700.

3.3.2. Lowest cost entry method

(a): In this method we start with the lowest cost position. Here it is (1,4) and (3,2) positions, allocate the maximum possible units to these positions, i.e., 8 units to the position (1,4) and 8 units to position (3,2), write them as 8(1) and 8(1) respectively, then strike off the other positions in row 1 and also in column 2, since all the available units are distributed to these positions.

Table 3.8

Depot Unit	B ₁	B ₂	B ₃	B ₄	Stock
A ₁	x	X	x	8(1)	0
A ₂		X			10
A ₃		8(1)			12
Requirement	6	0	9	7	22

(b) Consider the next higher cost positions, i.e., (1,1) and (3,4) positions, but the position (1,1) is already been struck off so we can't allocate any units to this position. Now allocate the maximum possible units to position (3,4), i.e., 7 units as required by the place and write it as 7(2). Hence the allocations in the column 4 is complete, so strike off the (2, 4) position.

Table 3.9

Depot Unit	B ₁	B ₂	B ₃	B ₄	Stock
A ₁	x	X	x	8(1)	0
A ₂		X		X	10
A ₃)		8(1		7(2)	5
Requirement	6	0	9	0	

(c) Again consider the next higher cost position, i.e., (1, 2) and (2, 2) positions, but these positions are already been struck off so we cannot allocate any units to these positions.

(d) Consider the next higher positions, i.e., (2, 3) and (3, 1) positions, allocate the maximum possible units to these positions, i.e., 9 units to position (2, 3) and 5 units to position (3, 1), write them as 9(4) and 5(4) respectively. In this way allocation in column 3 is complete so strike off the (3, 3) position.

Table 3.10

Depot Unit	B ₁	B ₂	B ₃	B ₄	Stock
A ₁	x	X	x	8 (1)	0
A ₂		X	9 (4)	X	1
A ₃	5(4)	8(1)	x	7(2)	0
Requirement	1	0	0	0	

(e) Now only the position (2, 1) remains and it automatically takes the allocation 1 to complete the total in this row, therefore, write it as 1(7). With the help of the above facts complete the allocation table as given below.

Table3. 11

Depot Unit	B ₁	B ₂	B ₃	B ₄	Stock
A ₁	x	X	x	8 (1)	0
A ₂	1(7)	X	9 (4)	X	1
A ₃	5(4)	8(1)	x	7(2)	20
Requirement	6	8	9	15	

From the above facts, calculate the cost of transportation as $8.1 + 1.7 + 9.4 + 5.4 + 8.1 + 7.2 = 8 + 7 + 36 + 20 + 8 + 14 = 93$ i.e., Rs.9300.

3.3.3. Vogel's Approximation Method

(a1): Write the difference of minimum cost and next to minimum cost against each row in the penalty column. (This difference is known as penalty). (a2) Write the difference of minimum cost and next to minimum cost against each column in the penalty row. (This difference is known as penalty).

Table 3.12

Depot Unit	B ₁	B ₂	B ₃	B ₄	Stock	Penalties
A ₁	(2)	(3)	(5)	(1)	8	(1)
A ₂	(7)	(3)	(4)	(6)	10	(1)
A ₃	(4)	(1)	(7)	(2)	20	(1)
Requirement	6	8	9	15	38	
Penalties	(2)	(2)	(1)	(1)		

(b) Identify the maximum penalties. In this case it is at column one and at column two. Consider any of the two columns, (here take first column) and allocate the maximum units to the place where the cost is minimum (here the position (1, 1) has minimum cost so allocate the maximum possible units, i.e., 6 units to this position). Now write the remaining stock in row one. After removing the first column and then by repeating the step (a), we obtain as follows:

Table 3.13

Depot Unit		B ₂	B ₃	B ₄	Stock	Penalties	
A ₁		(3)	(5)	(2)	2	(2)	
A ₂		(3)	(4)	(6)	10	(1)	
A ₃		(1)	(7)	(2)	20	(1)	
Requirement		8	9	15	32		
Penalties		(2)	(1)	(1)			

(c) Identify the maximum penalties. In this case it is at row one and at column two. Consider any of the two (let it be first row) and allocate the maximum possible units to the place where the cost is minimum (here the position (1, 4) has minimum cost so allocate the maximum possible units, i.e., 2 units to this position). Now write the remaining stock in column four. After removing the first row and by repeating the step (a), we obtain table 14 as given below.

Table 3.14

Depot Unit		B ₂	B ₃	B ₄	Stock	Penalties
A ₂		(3)	(4)	(6)	10	(1)
A ₃		(1)	(7)	(2)	20	(1)
Requirement		8	9	13	30	
Penalties		(2)	(3)	(4)		

(d) Identify the maximum penalties. In this case it is at column four. Now allocate the maximum possible units to the minimum cost position (here it is at (3, 4) position and allocate maximum possible units, i.e., 13 to this position). Now write the remaining stock in row three. After removing the fourth column and then by repeating the step (a) we obtain table 15 as given below.

Table 3.15

Depot Unit		B ₂	B ₃	B ₄	Stock	Penalties
A ₂		(3)	(4)		10	(1)
A ₃		(1)	(7)		7	(6)
Requirement		8	9			
Penalties		(2)	(3)			

(e) Identify the maximum penalties. In this case it is at row three. Now allocate the maximum possible units to the minimum cost position (here it is at (3, 2) position and allocate maximum possible units, i.e., 7 to this position). Now in order to complete the sum, (2, 2) position will take 1 unit and (2, 3) position will be allocated 9 units.

This completes the allocation and with the help of the above information draw table 16 as under.

Table 3.16

Depot Unit	B1	B2	B3	B4	Stock
	6 (2)			2(1)	8
A2		1(3)	9 (4)		10
A3		7 (1)		13 (2)	20
Requirement	6	8	9	15	38

From the above facts calculate the cost of transportation as $6 \times 2 + 2 \times 1 + 1 \times 3 + 9 \times 4 + 7 \times 1 + 13 \times 2 = 12 + 2 + 3 + 36 + 7 + 26 = 86$ i.e., Rs.8600.

Note:

After calculating the cost of transportation by the above three methods, one thing is clear that Vogel's approximation method gives an initial basic feasible solution which is much closer to the optimal solution than the other two methods. It is always worth while to spend some time finding a "good" initial solution because it can considerably reduce the total number of iterations required to reach an optimal solution.

3.4. Test for Optimization

In part (A) of this section we have learnt how to obtain an initial basic feasible solution. Solutions so obtained may be optimal or may not be optimal, so it becomes essential for us to test for optimization. For this purpose we first define non-degenerate basic feasible solution.

Definition: A basic feasible solution of an $(m \times n)$ transportation problem is said to be non degenerate if it has following two properties.

- Initial basic feasible solution must contain exactly $m + n - 1$ number of individual allocations.
- These allocations must be in independent positions. Independent positions of a set of allocations mean that it is always impossible to form any closed loop through these allocations.

The following theorem is also helpful in testing the optimality.

Theorem: If we have a B.F.S. consisting of $m + n - 1$ independent positive allocations and a set of arbitrary number u_i and v_j ($i=1, 2, \dots, m$; $j=1, 2, \dots, n$) such that $c_{rs} = u_r + v_s$ for all occupied cells (r,s) then the evaluation d_{ij} corresponding to each empty cell (i, j) is given by $d_{ij} = c_{ij} - (u_i + v_j)$. Algorithm for optimality test :

In order to test for optimality we should follow the procedure as given below:

Step 1: Start with B.F.S. consisting of $m+n-1$ allocations in independent positions.

Step 2: Determine a set of $m+n$ numbers u_i ($i=1,2,\dots,m$) and v_j ($j=1,2,\dots,n$) such that for each occupied cells (r,s) $c_{rs} = u_r + v_s$

Step 3: Calculate cell evaluations (unit cost difference) d_{ij} for each empty cell (i,j) by using the formula $d_{ij} = c_{ij} - (u_i + v_j)$

Step 4: Examine the matrix of cell evaluation d_{ij} for negative entries and conclude that

(i) If all $d_{ij} > 0$ $\Rightarrow$ Solution is optimal and unique.

(ii) If all $d_{ij} \geq 0$ $\wedge$ At least one $d_{ij} = 0$ $\Rightarrow$ Solution is optimal and alternate solution also exists.

(iii) If at least one $d_{ij} < 0$ $\Rightarrow$ Solution is not optimal.

If it is so, further improvement is required by repeating the above process. See step 5 and onwards

Step 5: (i) See the most negative cell in the matrix $[d_{ij}]$.

(ii) Allocate q to this empty cell in the final allocation table. Subtract and add the amount of this allocation to other corners of the loop in order to restore feasibility.

(iii) The value of q , in general is obtained by equating to zero the minimum of the allocations containing $-q$ (not $+q$) only at the corners of the closed loop.

(iv), Substitute the value of q and find a fresh allocation table.

Step 6: Again, apply the above test for optimality till you find all $d_{ij} \geq 0$

Computational demonstrations for optimality test

Consider the initial basic feasible solution as obtained by Vogel's approximation method in section (A) of this article table 17

Step 1: (i) In this table number of allocations = $3+4-1=6$.

(ii) All the positions of allocations are independent.

Step 2: Determine a set of $(m + n)$, i.e., $(3+4)$ numbers u_1, u_2, u_3 , and v_1, v_2, v_3 , and v_4 for each occupied cells.

For this consider the row or column in which the allocations are maximum (here, let us take first row)

Now, take u_1 as an arbitrary constant (say zero) then by using $c_{ij} = u_i + v_j$ try to find all u_i and v_j as

Table 2.17

	B ₁	B ₂	B ₃	B ₄	U _i
A ₁	2			1	0
A ₂		3	4		3
A ₃		1		2	1
V _j	2	0	1	1	

$$c_{11} = 2 = u_1 + v_1 = 0 + v_1 \quad v_1 = 2$$

$$\text{Then } c_{14} = 1 = u_1 + v_4 = 0 + v_4 \quad v_4 = 1$$

$$\text{Then } c_{34} = 2 = u_3 + v_4 = u_3 + 1 \quad u_3 = 1$$

$$\text{Then } c_{32} = 1 = u_3 + v_2 = 1 + v_2 \quad v_2 = 0$$

$$\text{Then } c_{22} = 3 = u_2 + v_2 = u_2 + 0 \quad u_2 = 3$$

$$\text{Then } c_{23} = 4 = u_2 + v_3 = 3 + v_3 \quad v_3 = 1$$

Thus $u_1 = 0, u_2 = 3, u_3 = 1$ and $v_1 = 2, v_2 = 0, v_3 = 1$ and $v_4 = 1$.

Step 4: Here all $d_{ij} > 0$ Solution obtained by Vogel's approximation method is an optimal solution

Example: For the transportation problem

Table 2.18

Warehouse Factory	W ₁	W ₂	W ₃	W ₄	Factory Capacity
F ₁	19	30	50	10	7
F ₂	70	30	40	60	9
F ₃	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Find the initial basic feasible solution by

1. Vogel's approximation method
2. NWC method
3. LC method
4. Test for optimality

Problems

1. The distances between Boston, Chicago, Dallas, Los Angeles, and Miami are given in the following table. Each city needs 40,000 kilowatt hours (kwh) of power, and Chicago, Dallas, and Miami are capable of producing 70,000 kwh. Assume that shipping 1,000 kwh over 100 miles costs \$4.00. From where should power be sent to minimize the cost of meeting each city's demand?

	Boston	Chicago	Dallas	LA	Miami
Chicago	983	0	1,205	2,112	1,390
Dallas	1,815	1,205	0	801	1,332
Miami	1,539	1,390	1,332	2,757	0

Problem 2

There are three warehouses at different cities: Detroit, Pittsburgh and Buffalo. They have 250, 130 and 235 tons of paper accordingly. There are four publishers in Boston, New York, Chicago and Indianapolis. They ordered 75, 230, 240 and 70 tons of paper to publish new books. There are the following costs in dollars of transportation of one ton of paper:

From \ To	Boston (BS)	New York (NY)	Chicago (CH)	Indianapolis (IN)
Detroit (DT)	15	20	16	21
Pittsburgh (PT)	25	13	5	11
Buffalo (BF)	15	15	7	17

We denote the cost of transportation from one city to another by C_{prefix} , for example cost from Buffalo to Chicago is $C_{\text{BF_CH}}$.

We have to find a plan that all orders will be performed and the transportation costs will be **minimized**.

Problem 3

The Ministry of Health maintains three blood banks in Khobar. In a particular day, five public and private hospitals have requested quantities of blood bags of type A-. The blood supplies from the banks, requested blood bags by the hospitals, and the cost of blood per bag are shown in the following table:

		Hospitals					
		1	2	3	4	5	Supply
Blood banks	1	8	6	3	7	5	20
	2	5	M	8	4	7	30
	3	6	3	9	6	8	30
Demand		25	25	20	10	20	

In this particular day, bank 2 cannot deliver blood to hospital 2; hence the cost of transportation per blood bag is set to M (very large number) to signify a very large cost.

Apply the least cost method to find a suitable starting basic solution for this problem

3.5. Assignment Problem

In real life, we are faced with the problem of allocating different personnel/ workers to different jobs. Not everyone has the same ability to perform a given job. Different persons have different abilities to execute the same task and these different capabilities are expressed in terms of cost/profit/time involved in executing a given job. Therefore, we have to decide: How to assign different workers to different jobs” so that, cost of performing such job is

minimized. And such assignment problems and methods of their solutions is the subject matter of this Unit.

Objectives: Main objective of assignment problem is to equip the learner to deal with following situation: a) Assignments of different jobs to different workers/different machines on one to one basis where time or cost of performing such job is given. b) Assignment of different personnel to different location or service station with the objective to maximize sales/profit/consumer reaches. c) To deal with a situation where number of jobs to be assigned do not match with number of machines/workers. d) To deal with a situation where some jobs can not be assigned to specific machines/workers.

The assignment problem is a special type of linear programming problem. We know that linear programming is an allocation technique to optimize a given objective. In linear programming we decide how to allocate limited resources over different activities so that, we maximize the profits or minimize the cost.

Similarly in assignment problem, assignees are being assigned to perform different task. For example, the assignees can be employees who need to be given work assignments, is a common application of assignment problem. However assignees need not be people. They could be machines, vehicle, plants, time slots etc. to be assigned different task.

3.6. Assumptions of an Assignment Problem:

An assignment problem must satisfy the following assumptions: 1. the number of assignees and number of task are the same (this number is denoted by n). 2. Each assignee is to be assigned to perform exactly one task. 3. Each task is to be performed by exactly one assignee. 4. There is a cost or profit associated with assignees performing different task. 5. The objective is to determine how all n assignment should be made to optimize the given pay offs which are expressed in terms of cost, time spent, distance, revenue earned, production obtained etc.

3.7. Areas of Use Assignment Problem:

There exist numbers of areas where assignment problem can be used. In fact, whenever we have to make an assignment on one to one basis assignment technique is used. For example,

assignments of different jobs to different workers, assignments of different machines to different workers, assignments of different salesmen to different sales, centre/location, assignments of different products to different machines, assigning different rooms to different managers.

Imagine, if in a printing press there is one machine and one operator is there to operate. How would you employ the worker? Your immediate answer will be, the available operator will operate the machine. Again suppose there are two machines in the press and two operators are engaged at different rates to operate them. Which operator should operate which machine for maximizing profit? Similarly, if there are n machines available and n persons are engaged at different rates to operate them. Which operator should be assigned to which machine to ensure maximum efficiency? While answering the above questions we have to think about the interest of the press, so we have to find such an assignment by which the press gets maximum profit on minimum investment. Such problems are known as "assignment problems".

⌚ Hungarian method cost assignment is a classical algorithm implemented in the D style. H.W.Kuhn published a pencil and paper version in 1955, which was followed by J.R.Munkres' execut-able version in 1957. The algorithm is sometimes referred to as the **"Hungarian method"**.

The method indicates an optimal assignment of a set of resources to a set of requirements, given a "cost" of each potential match. Examples might be the allocation of workers to tasks; the supply of goods by factories to warehouses; or the matching of brides with grooms. The function takes a cost matrix as argument and returns a Boolean assignment matrix result.

The following table shows an optimal assignment of factories F, G, H to ware-houses W, X, Y, given that the cost of transportation from F to W is 72 units, F to X is 99 units, $\cdots$, G to W is 23 units, $\cdots$ and so on.

	W	X	Y	
F	[72]	99	88	Minimum-cost assignment marked [.]:
G	23	30	[35]	Factory F supplies warehouse W,
H	51	[59]	84	.. G Y,
				.. H X.

Notice that if the problem requires maximizing a benefit, rather than minimizing Cost, then a negative cost matrix is used.

Technical notes:

Munkres' algorithm may be described in words as follows:

Step 0:

Ensure the costs matrix has at least as many rows as columns, by appending extra 0-item rows if necessary. Go to Step 1.

Step 1: Subtract the smallest item in each row from the row. Go to Step 2.

Step 2: Select a set of "independent" zeros in the matrix and mark them with a star (*).

To do this, star leading zeros in each row and column, ignoring rows and columns already containing stars; repeat this process until apart from ignored rows and columns, no more zeros remain. Go to Step 3.

Step 3: Draw a line through (cover) each column containing a starred zero. If all columns are covered, the starred zeros represent an optimal assignment. In this case, return a Boolean matrix with the positions of the stars, as result. Other-wise, go to Step 4.

Step 4: Find an uncovered zero. If there is none, go to Step 6 passing the smallest un-covered value as a parameter. Otherwise, mark the zero with a prime (') and call it P0. If there is a starred zero (S1) in the row containing P0, cover this row and uncover the column containing S1, then repeat Step 4. Otherwise, (if there is no starred zero in P0's row) go to Step 5.

Step 5: Find a path through alternating primes and stars. Starting with the uncovered prime (P0) found in Step 4, find a star S1 (if any) in its column. Then find a prime P2 (there must

be one) in S_1 's row, followed by a star S_3 (if any) in P_2 's column, $\dots$ and so on until a prime (P_n) is found that has no star in its column. In the series $P_0, S_1, P_2, S_3, \dots, P_n$, unstar each starred zero S_i and star each primed zero P_j . Finally, unprime all primed zeros in the matrix, un-

Cover all rows and columns. Go to Step 3.

Step 6: Add the minimum cost value passed from Step 4 to each twice-covered (row and column covered) item, and subtract it from each uncovered item. Preserving all stars, primes and covering lines, go to Step 4.

Example 3.2.2:

A job has four men available for work on four separate jobs. Only one man can work on any one job. The cost of assigning each man to each job is given in the following table. The objective is to assign men to jobs such that the total cost of assignment is minimum.

Table 3.2.1

Persons	Jobs			
	1	2	3	4
A	20	25	22	28
B	15	18	23	17
C	19	17	21	24
D	25	23	24	24

Solution:

Step 1

Identify the minimum element in each row and subtract it from every element of that row.

Table 3.2.2

Persons	Jobs			
	1	2	3	4
A	0	5	2	8
B	0	3	8	2
C	2	0	4	7
D	2	0	1	1

Step 2; identify the minimum element in each column and subtract it from every element of that column.

Table 3.2.3

Persons	Jobs			
	1	2	3	4
A	0	5	1	7
B	0	3	7	1
C	2	0	3	6
D	2	0	0	0

Step 3

Make the assignment for the reduced matrix obtain from **steps 1 and 2** in the following way:

- Examine the rows successively until a row with exactly one unmarked zero is found. Enclose this zero in a box as an assignment will be made there and cross (X) all other zeros appearing in the corresponding column as they will not be considered for future assignment. Proceed in this way until all the rows have been examined.

- b. After examining all the rows completely, examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single zero by putting square around it and cross out (X) all other assignments in that row proceed in this manner until all columns have been examined.
- c. Repeat the operations (a) and (b) successively until one of the following situations arises:

- All the zeros in rows/columns are either marked or crossed (X) and there is exactly one assignment in each row and in each column. In such a case optimum assignment policy for the given problem is obtained.

There may be some row (or column) without assignment, i.e. the total number of marked zeros is less than the order of the matrix. In such a case proceed to next step 4.

Table 3.2.4

Persons	Jobs			
	1	2	3	4
A	0	5	1	7
B	X	3	7	1
C	2	0	3	6
D	2	X	0	X

Step 4

Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix obtained from step 3 by adopting the following procedure:

- Mark all the rows that do not have assignments.
- Mark all the columns (not already marked) which have zeros in the marked rows.
- Mark all the rows (not already marked) that have assignments in marked columns.
- Repeat steps 4 (ii) and (iii) until no more rows or columns can be marked.
- Draw straight lines through all unmarked rows and columns.

You can also draw the minimum number of lines by inspection

Table

Table 3.2.5

Jobs				
Persons	1	2	3	4
A	0	5	1	7
B	8	3	7	1
C	2	0	3	6
D	4	8	0	8

Step 5

Select the smallest element from all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment.

Table 3.2.6

Jobs				
Persons	1	2	3	4
A	0	4	0	6
B	0	2	6	0
C	3	0	3	6
D	3	0	0	0

Go to **step 3** and repeat the procedure until you arrive at an optimum assignment.

Final Table

Table 3.2.7

Jobs				
Persons	1	2	3	4
A	0	4	8	6
B	0	2	6	0
C	3	0	3	6
D	3	8	0	8

Since the number of assignments is equal to the number of rows (& columns), this is the optimal solution.

The total cost of assignment = A1 + B4 + C2 + D3

Substitute the values from original table: $20 + 17 + 24 + 17 = 78$.

Example 3: Assignment Problem Dummy case

A contractor pays his subcontractors a fixed fee plus mileage for work performed. On a given day the contractor is faced with three electrical jobs associated with various projects. Given below are the distances between the subcontractors and the projects.

	Project			
	A	B	C	
Westside	50	36	16	
Federated		28	30	18
Goliath	35	32	20	
Universal		25	25	14

How should the contractors be assigned to minimize total costs?

Note: There are four subcontractors and three projects. We create a dummy project Dum, which will be assigned to one subcontractor (i.e. that subcontractor will remain idle)

Since the Hungarian algorithm requires that there be the same number of rows as columns, add a Dummy column so that the first tableau is (the smallest elements in each row are marked red):

	A	B	C	Dummy
Westside	50	36	16	0

Federated	28	30	18	0
Goliath	35	32	20	0
Universal	25	25	14	0

Subtract minimum number in each row from all numbers in that row. Since each row has a zero, we simply generate the original matrix (the smallest elements in each column are marked red). These yields:

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>
Westside	50	36	16	0
Federated	28	30	18	0
Goliath	35	32	20	0
Universal	25	25	14	0

Step 2: Subtract the minimum number in each column from all numbers in the column. For A it is 25, for B it is 25, for C it is 14, for Dummy it is 0. This yield:

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>
Westside	25	11	2	0
Federated	3	5	4	0
Goliath	10	7	6	0
Universal	0	0	0	0

Step 3: Draw the minimum number of lines to cover all zeroes (called minimum cover). Although one can "eyeball" this minimum, use the following algorithm. If a "remaining" row has only one zero, draw a line through the column. If a remaining column has only one zero in it, draw a line through the row. Since the number of lines that cover all zeros is $2 < 4$ (# of rows), the current solution is not optimal.

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>
Westside	25	11	2	0
Federated	3	5	4	0
Goliath	10	7	6	0
Universal	0	0	0	0

Step 4: The minimum uncovered number is 2 (circled).

Step 5: Subtract 2 from uncovered numbers; add 2 to all numbers at line intersections; leave all other numbers intact. This gives:

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>
Westside	23	9	0	0
Federated	1	3	2	0
Goliath	8	5	4	0
Universal	0	0	0	2

Step 3: Draw the minimum number of lines to cover all zeroes. Since 3 (# of lines) $<$ 4 (# of rows), the current solution is not optimal.

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>
Westside	23	9	0	0
Federated	1	3	2	0
Goliath	8	5	4	0
Universal	0	0	0	2

Step 4: The minimum uncovered number is 1 (circled).

Step 5: Subtract 1 from uncovered numbers. Add 1 to numbers at intersections. Leave other numbers intact. This gives:

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>	
Westside	23	9	0	0	1
Federated	0	2	1	0	
Goliath	7	4	3	0	
Universal	0	0	0	3	

Find the minimum cover:

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>
Westside	23	9	0	1
Federated	0	2	1	0
Goliath	7	4	3	0
Universal	0	0	0	3

Step 4: The minimum number of lines to cover all 0's is four. Thus, the current solution is optimal (minimum cost) assignment.

Find the minimum cover:

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>
Westside	23	9	0	1
Federated	0	2	1	0
Goliath	7	4	3	0
Universal	0	0	0	3

Step 4: The minimum number of lines to cover all 0's is four. Thus, the current solution is optimal (minimum cost) assignment.

The optimal assignment occurs at locations of zeros such that there is exactly one zero in each row and each column:

	<u>A</u>	<u>B</u>	<u>C</u>	<u>Dummy</u>
Westside	23	9	0	1
Federated	0	2	1	0
Goliath	7	4	3	0
Universal	0	0	0	3

The optimal assignment is (go back to the original table for the distances):

<u>Subcontractor</u>	<u>Project</u>	<u>Distance</u>
Westside	C	16
Federated	A	28
Universal	B	25
Goliath	(unassigned)	
Total Distance = 69 miles		

Problem 1

Suppose an auto company has three plants in cities A, B and C and two major distribution centers in D and E. The capacities of the three plants during the next quarter are 1000, 1500 and 1200 cars. The quarterly demands of the two distribution centers are 2300 and 1400 cars. The transportation costs (which depend on the mileage, transport company etc) between the plants and the distribution centers is as follows:

Cost Table	Dist Center D	Dist Center E
Plant A	80	215
Plant B	100	108
Plant C	102	68

Which plant should supply how many cars to which outlet so that the total cost is Minimum?

Problem 2

Three men are to be given 3 jobs and it is assumed that a person is fully capable of doing a job independently. The following table gives an idea of that cost incurred to complete each job by each person:

Jobs → Men ↓	J ₁	J ₂	J ₃	Supply
M ₁	20	28	21	1
M ₂	15	35	17	1
M ₃	8	32	20	1
Demand	1	1	1	

Required: Assign the right person to the right job in a cost minimizing manner

Problem 3

A typical assignment problem, presented in the classic manner, is shown in Fig. below. Here there are five machines to be assigned to five jobs. The numbers in the matrix indicate the cost of doing each job with each machine. Jobs with costs of M are disallowed assignments. The problem is to find the minimum cost matching of machines to jobs.

	J1	J2	J3	J4	J5
M1	M	8	6	12	1
M2	15	12	7	M	10
M3	10	M	5	14	M
M4	12	M	12	16	15
M5	18	17	14	M	13

Matrix model of the assignment problem

The network model is in Fig. 13. It is very similar to the transportation model except the external flows are all +1 or -1. The only relevant parameter for the assignment model is arc cost (not shown in the figure for clarity) ; all other parameters should be set to default values. The assignment network also has the bipartite structure.

The solution to the assignment problem as shown in Fig. 14 has a total flow of 1 in every column and row, and is the assignment that minimizes total cost.

	J1	J2	J3	J4	J5
M1	0	0	0	0	1
M2	0	0	1	0	0
M3	0	0	0	1	0
M4	1	0	0	0	0
M5	0	1	0	0	0

Problem 4

Zain being a new mobile operator in the Kingdom has to construct its mobile network. To serve its anticipated customers in King Abdul Azziz airbase, KFUPM, and Saudi Aramco, Zain operation department has, as an initial stage, decided to erect 4 mobile towers in Dhahran area. Four construction companies have applied for the construction tasks. Since the constructions have to be started at the same time, a construction company can work on only one construction task. The construction costs in hundred thousand riyals are as follows:

		Company			
		1	2	3	4
Tower	1	4	6	5	5
	2	7	4	5	6
	3	4	7	6	4
	4	5	3	4	7

Use the appropriate method to find the best assignment and calculate the corresponding total cost

🔧 Summery

Transportation problem is a special class of linear programming problem in which the objective is to “transport” or distribute a single commodity from several „sources” (also called origins or supplies or capacity centers) to different „destinations” (also such as demands or requirement (enters) at a minimum total cost. One important application of linear programming is in the area of physical distribution (transportation) of goods and services from several supply origins to several demand destinations.

Assignment is allocating different personnel/ workers to different jobs. Not everyone has the same ability to perform a given job. Different persons have different abilities to execute the same task and these different capabilities are expressed in terms of cost/profit/time involved in executing a given job. Therefore, we have to decide: How to assign different workers to different jobs” so that, cost of performing such job is minimized

Self Test Exercise 3

Part I: Write “True” if the statement is correct and “False” if the statement is incorrect.

1. The total supply must equal total demand in a transportation problem in order to solve it by the transportation algorithm.
2. In a linear programming formulation of the assignment problem, the RHSV of all constraint is greater than one.
3. The least cost method is one means of testing the optimality of transportation problem.

Transportation and Assignment are the same.

Part III: show all the necessary steps.

- I.** Develop an initial feasible solution using NWCM

Table: Unbalanced transportation table

	<i>R</i>	<i>S</i>	<i>T</i>	<i>Supply</i>
<i>A</i>	1	2	3	100
<i>B</i>	4	1	5	110
<i>Demand</i>	80	120	60	210 260

2. Determine an initial feasible solution to the following transportation problem using
- NWCM
 - LCM,
 - VAM,
 - Determine its cost, &
 - Test its optimality.

		<i>Destination</i>				
		D ₁	D ₂	D ₃	D ₄	<i>Supply</i>
<i>Source</i>	<i>A</i>	11	13	17	14	250
	<i>B</i>	16	18	14	10	300
	<i>C</i>	21	24	13	10	400
	<i>Demand</i>	200	225	275	250	

UNIT FOUR: DECISION THEORY

4. Introduction

Dear learner, the success or failure of an organization or individual depends to a large extent on the ability of making appropriate decisions. Making of a decision requires enumeration of feasible and visual alternatives (courses of actions or strategies), the projection of consequences associated with different alternatives and a measure of effectiveness (or an objectives) by which the most preferred alternative is identified. Decision theory provides an analytical and systematic approach to the study of decision making where in data concerning the occurrences of different outcomes may be evaluated to enable the decision maker to identify suitable alternative course of action.

Learning Objectives: - At the end of this unit, you will be able to:-

- *Describe what decision making is;*
- *Discuss about the decision theory;*
- *Use decision trees in decision making;*
- *Explain the different situations under which decision will be made.*

4.1. Decision Theory

Dear learners, how are you going to define the term decision making? Use the space below to express your feelings.

Typically, personal and professional decisions can be made with some difficulty. Either the best course of action is clear or the varieties of the decision are not significant enough to require a great amount of attention. Occasionally, decisions arise where the path is not clear and it is necessary to take substantial time and effort in devising a systematic method of analyzing the various courses of action. With decisions under uncertainty, the decision maker should:

1. Take an inventory of all viable options available for gathering information, for experimentation and for action
2. List all events that may occur
3. Arrange all pertinent information and choices/assumptions made
4. Rank the consequences resulting from the various courses of action
5. Determine the probability of an uncertain event occurring.

Upon systematically describing the problem and recording all necessary data, judgments, and preferences, the decision maker should synthesize the information set before using the most appropriate decision rules. Decision rules prescribe how an individual faced with a decision under uncertainty should go about choosing a course of action consistent with the individual's basic judgments and preferences.

When a decision maker should choose one possible actions, the ultimate consequences of some, if not all of these actions will generally depend on uncertain events and future actions extending indefinitely far into the future. The uncertainty is specially expressed in agriculture. Sahin et al. (2008) determine the cattle fattening breed, which maximizes the net profit for the producers under risk and uncertainties. The Wald's, Hurwicz's, Maxi-max, Savage's, Laplace's and Utility criterions were used. On the other hand the decision on which crops to include in crop rotation is one of the most important decisions in field crop farm management. Agronomic, economical and market information about each individual crop constitutes an informative basis for decision-making. There is a significant amount of valuable agronomic and market information already available on main crop production, including oil crops (Rozman et al., 2006). However, the potential for a wider range of alternative crops, including oil pumpkin (Bavec and Bavec, 2006), should be evaluated in order to determine their break-crop characteristics and the benefits and challenges which they bring to systems (Robson et al., 2002). According to Lampkin and Measures (1999), the economics of oil pumpkin depends on market price, therefore enquires with potential buyers should be undertaken. However, recent farm management research has also shown oil pumpkin production can be financially feasible assuming that the pumpkin oil can be successfully sold. Pažek (2003) and Pažek et al. (2005) conducted a financial and economical analysis of farm product processing on Slovene farms using a simulation - modeling approach that included

also pumpkin oil production. In agriculture there is a lack of studies that observe the application of criteria in the situation under uncertainty. From this reason in the paper five decision rules (criteria) commonly used in decision process under uncertainty were presented and applied in the case study of production and processing of oil pumpkin:

4.2. Decision Making Criteria

The following decision criteria are used in making decision

4.2.1. Wald's Maxi-min criterion

4.2.2. Hurwicz's criterion

4.2.3. Maxi-max criterion

4.2.4. Savage's mini-max regret criterion

4.2.5. Laplace's insufficient reason criterion.

4.2.1. Wald's Maxi -min Criterion

The decision-theoretic view of statistics advanced by Wald had an obvious interpretation in terms of decision-making under complete ignorance, in which the maxi min strategy was shown to be a best response against nature's mini max strategy. Wald's criterion is extremely conservative even in a context of complete ignorance, though ultra-conservatism may sometimes make good sense (Wen and Iwamura, 2008). The Maxi min criterion is a pessimistic approach. It suggests that the decision maker examines only the minimum payoffs of alternatives and chooses the alternative whose outcome is the least bad. This criterion appeals to the cautious decision maker who seeks assurance that in the event of an unfavorable outcome minimum payoffs may have a higher probability of occurrence or the lowest payoff may lead to an extremely unfavorable outcome.

4.2.2. Hurwicz's Optimism – Pessimism Criterion

The most well-known criterion is the Hurwicz criterion, suggested by Leonid Hurwicz in 1951, which selects the minimum and the maximum payoff to each given action x . The Hurwicz criterion attempts to find a middle ground between the extremes posed by the optimist and pessimist criteria. Instead of assuming total optimism or pessimism, Hurwicz

incorporates a measure of both by assigning a certain percentage weight to optimism and the balance to pessimism. However, this approach attempts to strike a balance between the maxi-max and maxi-min criteria. It suggests that the minimum and maximum of each strategy should be averaged using a and $1 - a$ as weights. a represents the index of pessimism and the alternative with the highest average selected. The index a reflects the decision maker's attitude towards risk taking. A cautious decision maker will set $a = 1$ which reduces the Hurwicz criterion to the maxi-min criterion. An adventurous decision maker will set $a = 0$ which reduces the Hurwicz criterion to the maxi-max criterion.

The Hurwicz criterion attempts to find a middle ground between the extremes posed by the optimist and pessimist criteria. Instead of assuming total optimism or pessimism, Hurwicz incorporates a measure of both by assigning a certain percentage weight to optimism and the balance to pessimism.

A weighted average can be computed for every action alternative with an alpha-weight α , called the coefficient of realism. "Realism" here means that the unbridled optimism of Maxi-max is replaced by an attenuated optimism as denoted by the α . Note that $0 \leq \alpha \leq 1$. Thus, a better name for the coefficient of realism is coefficient of optimism. $\alpha = 1$ denotes absolute optimism (Maxi-max) while $\alpha = 0$ indicates absolute pessimism (Maxi-min). α is selected subjectively by the decision maker.

Selecting a value for α simultaneously produces a *coefficient of pessimism* $1 - \alpha$, which reflects the decision maker's aversion to risk. A Hurwicz weighted average H can now be computed for every action alternative A_i in A as follows:

$H(A_i) = \alpha$ (row maximum) + $(1 - \alpha)$ (row minimum) - for positive-flow payoffs (profits, revenues)

$H(A_i) = \alpha$ (row minimum) + $(1 - \alpha)$ (row maximum) - for negative-flow payoffs (costs, losses)

Hurwicz decision rule is followed:

1. Select a coefficient of optimism value α .
2. For every action alternative compute its Hurwicz weighted average H .
3. Choose the action alternative with the best H as the chosen decision ("Best" means $\text{Max } \{H\}$ for positive-flow payoffs, and $\text{Min } \{H\}$ for negative-flow payoffs).

4.2.3. Hurwicz's Optimism – Pessimism Criterion

The most well-known criterion is the Hurwicz criterion, suggested by Leonid Hurwicz in 1951, which selects the minimum and the maximum payoff to each given action x . The Hurwicz criterion attempts to find a middle ground between the extremes posed by the optimist and pessimist criteria. Instead of assuming total optimism or pessimism, Hurwicz incorporates a measure of both by assigning a certain percentage weight to optimism and the balance to pessimism. However, this approach attempts to strike a balance between the maxi-max and maxi-min criteria. It suggests that the minimum and maximum of each strategy should be averaged using a and $1 - a$ as weights. a represents the index of pessimism and the alternative with the highest average selected. The index a reflects the decision maker's attitude towards risk taking. A cautious decision maker will set $a = 1$ which reduces the Hurwicz criterion to the maxi-min criterion. An adventurous decision maker will set $a = 0$ which reduces the Hurwicz criterion to the maxi-max criterion.

The Hurwicz criterion attempts to find a middle ground between the extremes posed by the optimist and pessimist criteria. Instead of assuming total optimism or pessimism, Hurwicz incorporates a measure of both by assigning a certain percentage weight to optimism and the balance to pessimism.

A weighted average can be computed for every action alternative with an alpha-weight α , called the coefficient of realism. "Realism" here means that the unbridled optimism of Maxi-max is replaced by an attenuated optimism as denoted by the α . Note that $0 \leq \alpha \leq 1$. Thus, a better name for the coefficient of realism is coefficient of optimism. $\alpha = 1$ denotes absolute optimism (Maxi-max) while $\alpha = 0$ indicates absolute pessimism (Maxi-min). α is selected subjectively by the decision maker.

Selecting a value for α simultaneously produces a *coefficient of pessimism* $1 - \alpha$, which reflects the decision maker's aversion to risk. A Hurwicz weighted average H can now be computed for every action alternative A_i in A as follows:

$H(A_i) = \alpha (\text{row maximum}) + (1 - \alpha) (\text{row minimum})$ - for positive-flow payoffs (profits, revenues)

$H(A_i) = \alpha$ (row minimum) + $(1 - \alpha)$ (row maximum) - for negative-flow payoffs (costs, losses)

Hurwicz decision rule is followed:

1. Select a coefficient of optimism value α .
2. For every action alternative compute its Hurwicz weighted average H .
3. Choose the action alternative with the best H as the chosen decision ("Best" means $\text{Max } \{H\}$ for positive-flow payoffs, and $\text{Min } \{H\}$ for negative-flow payoffs).

4.2.4. Maxi-max Criterion

The Maxi-max criterion is an optimistic approach. It suggests that the decision maker examine the maximum payoffs of alternatives and choose the alternative whose outcome is the best. This criterion appeals to the adventurous decision maker who is attracted by high payoffs. This approach may also Appeal to a decision maker who likes to gamble and who is in the position to withstand any losses without substantial inconvenience.

It is possible to model the optimist profile with the Maxi-max decision rule (when the payoffs are positive-flow rewards, such as profits or revenue. When payoffs are given as negative-flow rewards, such as costs, the optimist decision rule is Mini-min Note that negative-flow rewards are expressed with positive numbers.)

Maxi-max decision rule is followed:

1. For each action alternative (matrix row) determine the maximum payoff possible.
2. From these maxima, select the maximum payoff. The action alternative leading to this payoff is the chosen decision.

4.2.5. Savage's Mini-max Regret

The Savage Mini-max Regret criterion examines the regret, opportunity cost or loss resulting when a particular situation occurs and the payoff of the selected alternative is smaller than the payoff that could have been attained with that particular situation. The regret corresponding to a particular payoff X_{ij} is defined as $R_{ij} = X_j(\text{max}) - X_{ij}$ where $X_j(\text{max})$ is the maximum payoff attainable under the situation S_j . This definition of regret allows the decision maker to transform the payoff matrix into a regret matrix. The mini-max criterion suggests that the decision maker looks at the maximum regret of each strategy and selects the

one with the smallest value. This approach appeals to cautious decision makers who want to ensure that the selected alternative does well when compared to other alternatives regardless of the situation arising. It is particularly attractive to a decision maker who knows that several competitors face identical or similar circumstances and who is aware that the decision maker's performance will be evaluated in relation to the competitors. This criterion is applied to the same decision situation and transforms the payoff matrix into a regret matrix.

The Mini-max Regret criterion focuses on avoiding the worst possible consequences that could result when making a decision. Although regret is an emotional state (a psychological sense of loss) which, being subjective can be problematic to assess accurately, the assumption is made that regret is quantifiable in direct (linear) relation to the rewards R_{ij} expressed in the payoff matrix. This means that an actual loss of, say, an euro (an accounting loss) will be valued exactly the same as a failure to take advantage of the opportunity to gain an additional euro (an opportunity loss, which is disregarded in financial accounting). In other words, the Mini-max Regret criterion views actual losses and missed opportunities as equally comparable.

Regret is defined as the opportunity loss to the decision maker if action alternative A_i is chosen and state of nature S_j happens to occur. Opportunity loss (OL) is the payoff difference between the best possible outcome under S_j and the actual outcome resulting from choosing A_i given that S_j occurs. Thus, if the decision alternative secures the best possible payoff for a given state of nature, the opportunity loss is defined to be zero. Otherwise, the opportunity loss will be a positive quantity. Negative opportunity losses are not defined. Savage's Mini-max Regret criterion is formally defined as:

$OL_{ij} = (\text{column } j \text{ maximum payoff}) - R_{ij}$ - for positive-flow payoffs (profits, income)

$OL_{ij} = R_{ij} - (\text{column } j \text{ minimum payoff})$ - for negative-flow payoffs (costs)

Where R_{ij} is the payoff (reward) for row i and column j of the payoff matrix R .

Opportunity losses are defined as nonnegative numbers. The best possible OL is zero (no regret), and the higher OL value, the greater the regret.

Mini-max Regret decision rule is defined as:

1. Convert the payoff matrix $R = \{R_{ij}\}$ into an opportunity loss matrix $OL = \{OL_{ij}\}$.
2. Apply the mini-max rule to the OL matrix.

4.2.6. Laplace's Criterion

The Laplace's insufficient reason criterion postulates that if no information is available about the probabilities of the various outcomes, it is reasonable to assume that they are likely equally. Therefore, if there are n outcomes, the probability of each is $1/n$. This approach also suggests that the decision maker calculate the expected payoff for each alternative and select the alternative with the largest value. The use of expected values distinguishes this approach from the criteria of using only extreme payoffs. This characteristic makes the approach similar to decision making under risk. The Laplace's criterion is the first to make explicit use of probability assessments regarding the likelihood of occurrence of the states of nature. As a result, it is the first elementary model to use all of the information available in the payoff matrix.

The Laplace's argument makes use of Jakob Bernoulli's **Principle of Insufficient Reason**. The principle, first announced in Bernoulli's posthumous masterpiece, *Ars Conjectandi* (*The Art of Conjecturing*, 1713), states that "in the absence of any prior knowledge, we should assume that the events have equal probability". It means that the events are mutually exclusive and collectively exhaustive. Laplace posits that, to deal with uncertainty rationally, probability theory should be invoked. This means that for each state of nature (S_j in S), the decision maker should assess the probability of p_j that S_j will occur. This can always be done - theoretically, empirically or subjectively. Laplace decision rule is followed:

1. Assign $p_j = P(S_j) = 1/n$ to each S_j in S , for $j = 1, 2, \dots, n$.
2. For each A_i (payoff matrix row), compute its expected value: $E(A_i) = \sum_j p_j (R_{ij})$.
for $i = 1, 2, \dots, m$. Since p_j is a constant in Laplace, $E(A_i) = \sum_j p_j (R_{ij}) = p_j \sum_j R_{ij}$.
3. Select the action alternative with the best $E(A_i)$ as the optimal decision. "Best" means max for positive-flow payoffs (profits, revenues) and min for negative-flow payoffs (costs)

Examples

A tool commonly used to display information needed for the decision process is a payoff matrix or decision table. The table shown below is an example of a payoff matrix. The A 's stand for the alternative actions available to the decision maker. These actions represent the controllable variables in the system. The uncertain events or states of nature are represented

by the S's. Each S has an associated probability of its occurrence, denoted P. (However, the only decision rule that makes use of the probabilities is the Laplace criterion.) The payoff is the numerical value associated with an action and a particular state of nature. This numerical value can represent monetary value, utility, or both. This type of table will be used to illustrate each type of decision rule.

Table 4.1

Actions\States	S1 (P=.25)	S2 (P=.25)	S3 (P=.25)	S4 (P=.25)
A1	20	60	-60	20
A2	0	20	-20	20
A3	50]	-20	-80	20

This generic/hypothetical example illustrates 3 different actions that can be taken, and 4 different possible, uncertain states of nature with their respective payoffs.

I. Hurwicz criterion.

This approach attempts to strike a balance between the maxi max and maxi min criteria. It suggests that the minimum and maximum of each strategy should be averaged using a and $1 - a$ as weights. a represents the index of pessimism and the alternative with the highest average is selected. The index a reflects the decision maker's attitude towards risk taking. A cautious decision maker will set $a = 1$ which reduces the Hurwicz criterion to the maxi min criterion. An adventurous decision maker will set $a = 0$ which reduces the Hurwicz criterion to the maxi max criterion. A decision table illustrating the application of this criterion (with $a = .5$) to a decision situation is shown below.

Table 4.2:

Actions\States	S1	S2	S3	S4	$a = .5$
A1	20	60	-60	20	0
A2	0	20	-20	20	0
A3	50	-20	-80	20	-15

Hurwicz criterion illustration ($\alpha = .5$); Here the probability of each state is not considered; results in a tie between the first two alternatives.

II. Laplace Insufficient Reason Criterion.

The Laplace insufficient reason criterion postulates that if no information is available about the probabilities of the various outcomes, it is reasonable to assume that they are equally likely. Therefore, if there are n outcomes, the probability of each is $1/n$. This approach also suggests that the decision maker calculate the expected payoff for each alternative and select the alternative with the largest value. The use of expected values distinguishes this approach from the criteria that use only extreme payoffs. This characteristic makes the approach similar to decision making under risk. A table illustrates this criterion below.

Table 4.3:

Actions\States	S1 (P=.25)	S2 (P=.25)	S3 (P=.25)	S4 (P=.25)	Expected Payoff:
A1	20	60	-60	20	0
A2	0	20	-20	20	5
A3	50	-20	-80	20	-7.5

Laplace insufficiency illustration; Second alternative wins when expected payoff is calculated between equitable states

III. Maxi-max criterion.

The maxi-max criterion is an optimistic approach. It suggests that the decision maker examine the maximum payoffs of alternatives and choose the alternative whose outcome is the best. This criterion appeals to the adventurous decision maker who is attracted by high payoffs. This approach may also appeal to a decision maker who likes to gamble and who is in the position to withstand any losses without substantial inconvenience. See the table below for an illustration of this criterion.

Table 4.4:

Actions\States	S1	S2	S3	S4	Max Payoff
A1	20	60	-60	20	60
A2	0	20	-20	20	20
A3	50	-20	-80	20	50

Maxi-max illustration; First alternative wins

IV. Maxi-min criterion.

The maxi-min criterion is a pessimistic approach. It suggests that the decision maker examine only the minimum payoffs of alternatives and choose the alternative whose outcome is the least bad. This criterion appeals to the cautious decision maker who seeks to ensure that in the event of an unfavorable outcome, there is at least a known minimum payoff. This approach may be justified because the minimum payoffs may have a higher probability of occurrence or the lowest payoff may lead to an extremely unfavorable outcome. This criterion is illustrated in the table below.

Table 4.5:

Actions\States	S1	S2	S3	S4	Min payoff
A1	20	60	-60	20	-60
A2	0	20	-20	20	-20
A3	50	-20	-80	20	-80

Maxi-min illustration, Second alternative wins.

V. Savage mini-max regret criterion.

The Savage mini-max regret criterion examines the regret, opportunity cost or loss resulting when a particular situation occurs and the payoff of the selected alternative is smaller than the payoff that could have been attained with that particular situation. The regret corresponding to a particular payoff X_{ij} is defined as $R_{ij} = X_j(\max) - X_{ij}$ where $X_j(\max)$ is

the maximum payoff attainable under the situation S_j . This definition of regret allows the decision maker to transform the payoff matrix into a regret matrix. The mini-max criterion suggests that the decision maker look at the maximum regret of each strategy and select the one with the smallest value. This approach appeals to cautious decision makers who want to ensure that the selected alternative does well when compared to other alternatives regardless of what situation arises. It is particularly attractive to a decision maker who knows that several competitors face identical or similar circumstances and who is aware that the decision maker's performance will be evaluated in relation to the competitors. This criterion is applied to the same decision situation and transforms the payoff matrix into a regret matrix. This is shown below.

Table 4.6

Actions\States	R1	R2	R3	R4	Max Regret
A1	30	0	40	0	40
A2	50	40	0	0	50
A3	0	80	60	0	80

Mini-max illustration, first alternative wins.

4.3. Decision Tree

Is a tree like structure used for making different decision within the organization?

Decision tree example

A company is trying to decide whether to bid for a certain contract or not. They estimate that merely preparing the bid will cost Birr 10,000. If their company bid then they estimate that there is a 50% chance that their bid will be put on the "short-list", otherwise their bid will be rejected.

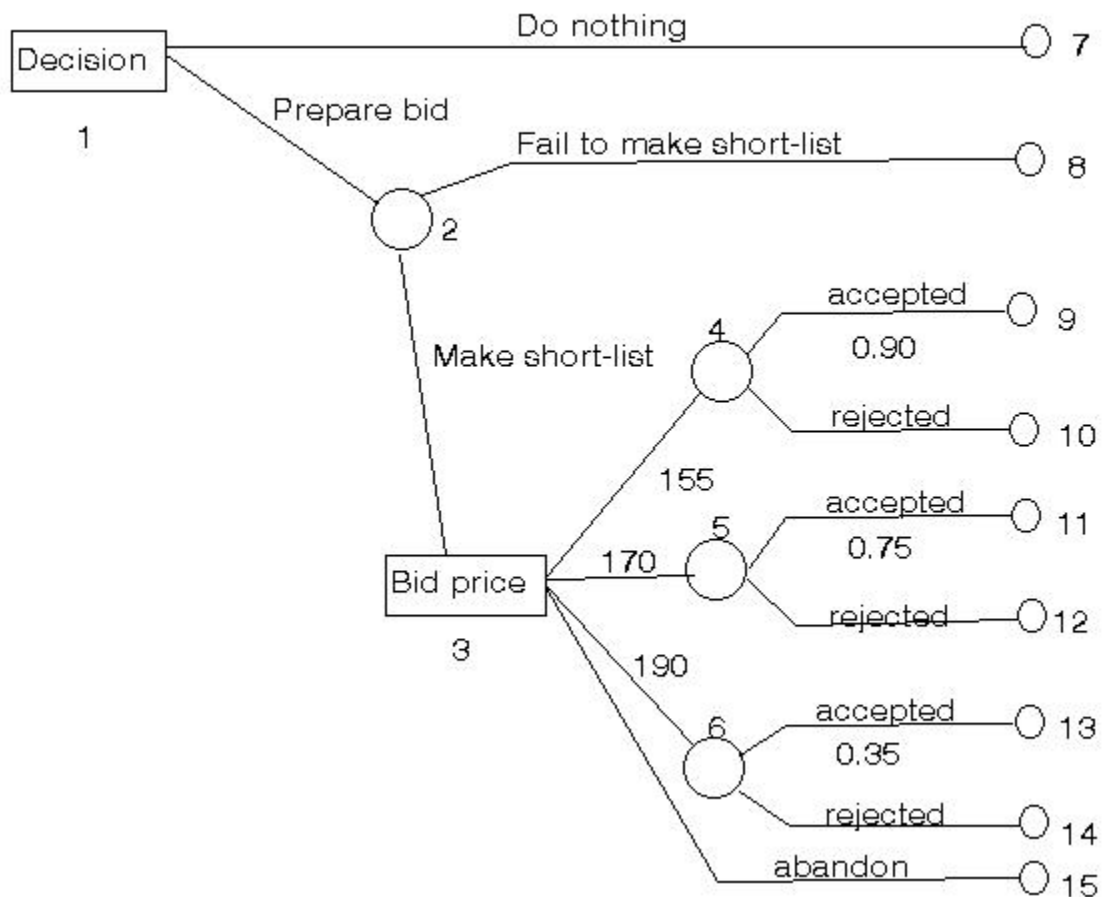
Once "short-listed" the company will have to supply further detailed information (entailing costs estimated at Birr 5,000). After this stage their bid will either be accepted or rejected.

The company estimates that the labor and material costs associated with the contract are Birr 127,000. They are considering three possible bid prices, namely Birr 155,000, birr 170,000 and birr 190,000. They estimate that the probability of these bids being accepted (once they have been short-listed) is 0.90, 0.75 and 0.35 respectively.

What should the company do and what is the expected monetary value of your suggested course of action?

Solution

The decision tree for the problem is shown on figure 3.1 below



Below we carry out step 1 of the decision tree solution procedure which (for this example) involves working out the total profit for each of the paths from the initial node to the terminal node (all figures in '000).

Step 1

- path to terminal node 7 - the company do nothing

Total profit = 0

- path to terminal node 8 - the company prepare the bid but fail to make the short-list

Total cost = 10 Total profit = -10

- path to terminal node 9 - the company prepare the bid, make the short-list and their bid of birr 155 is accepted

Total cost = 10 + 5 + 127 Total revenue = 155 Total profit = 13

- path to terminal node 10 - the company prepare the bid, make the short-list but their bid of birr 155 is unsuccessful

Total cost = 10 + 5 Total profit = -15

- path to terminal node 11 - the company prepare the bid, make the short-list and their bid of birr 170 is accepted

Total cost = 10 + 5 + 127 Total revenue = 170 Total profit = 28

- path to terminal node 12 - the company prepare the bid, make the short-list but their bid of Birr 170 is unsuccessful

Total cost = 10 + 5 Total profit = -15

- path to terminal node 13 - the company prepare the bid, make the short-list and their bid of birr 190 is accepted

Total cost = 10 + 5 + 127 Total revenue = 190 Total profit = 48

- path to terminal node 14 - the company prepare the bid, make the short-list but their bid of birr 190 is unsuccessful

Total cost = 10 + 5 Total profit = -15

- path to terminal node 15 - the company prepare the bid and make the short-list and then decide to abandon bidding (an implicit option available to the company)

Total cost = 10 + 5 Total profit = -15

Hence we can arrive at the table below indicating for each branch the total profit involved in that branch from the initial node to the terminal node.

Terminal node Total profit (birr)

7	0
8	-10
9	13
10	-15
11	28
11	-15
13	48
14	-15
15	-15

We can now carry out the second step of the decision tree solution procedure where we work from the right-hand side of the diagram back to the left-hand side.

Step 2

Consider chance node 4 with branches to terminal nodes 9 and 10 emanating from it. The expected monetary value for this chance node is given by $0.90(13) + 0.10(-15) = 10.2$

Similarly the EMV for chance node 5 is given by $0.75(28) + 0.25(-15) = 17.25$

The EMV for chance node 6 is given by $0.35(48) + 0.65(-15) = 7.05$

Hence at the bid price decision node we have the four alternatives

(1) Bid Birr 155 $EMV = 10.2$

(2) Bid Birr 170 $EMV = 17.25$

(3) Bid birr 190 $EMV = 7.05$

(4) Abandon the bidding $EMV = -15$

Hence the best alternative is to bid Birr 170 leading to an EMV of 17.25

Hence at chance node 2 the EMV is given by $0.50(17.25) + 0.50(-10) = 3.625$

Hence at the initial decision node we have the two alternatives

(1) Prepare bid $EMV = 3.625$

(2) Do nothing $EMV = 0$

Hence the best alternative is to prepare the bid leading to an EMV of Birr 3625. In the event that the company is short-listed then (as discussed above) it should bid birr 170,000.

Example 2

A householder is currently considering insuring the contents of his house against theft for one year. He estimates that the contents of his house would cost him £20,000 to replace.

Local crime statistics indicate that there is a probability of 0.03 that his house will be broken into in the coming year. In that event his losses would be 10%, 20%, or 40% of the contents with probabilities 0.5, 0.35 and 0.15 respectively.

An insurance policy from company A costs £150 a year but guarantees to replace any losses due to theft.

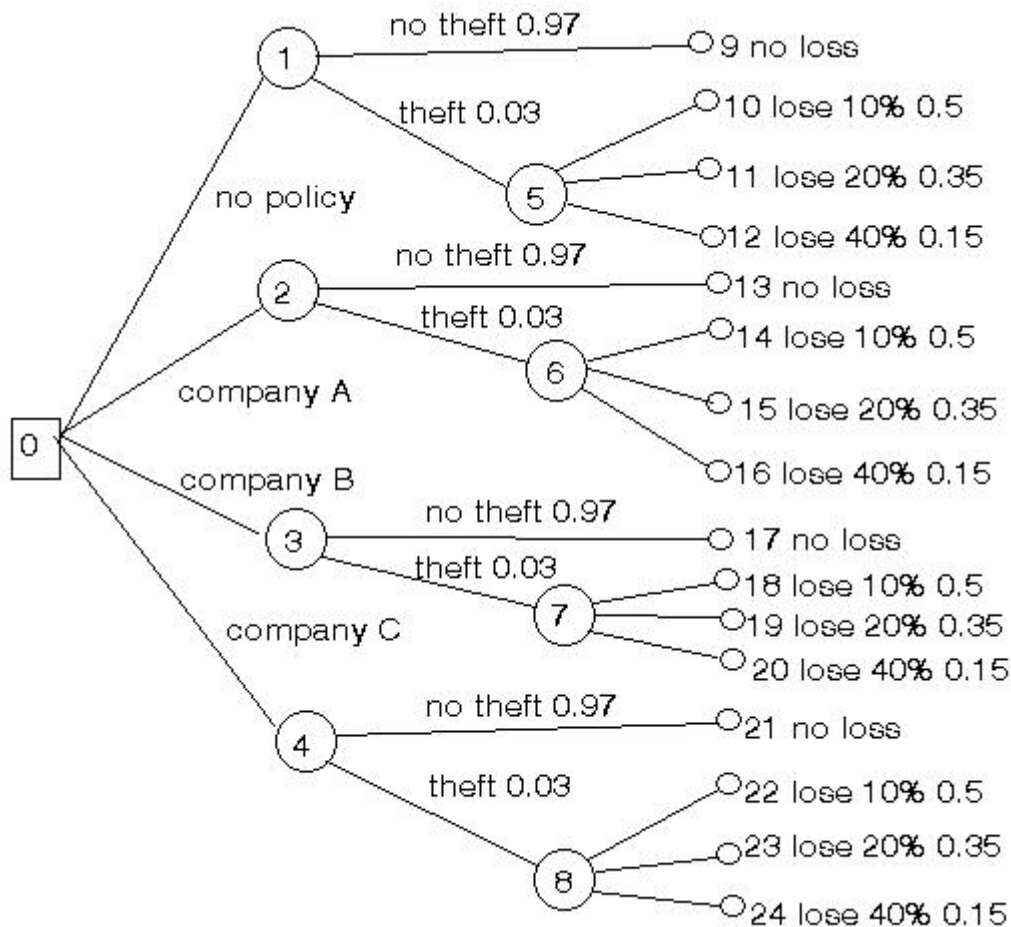
An insurance policy from company B is cheaper at £100 a year but the householder has to pay the first £x of any loss himself. An insurance policy from company C is even cheaper at £75 a year but only replaces a fraction (y%) of any loss suffered.

Assume that there can be at most one theft a year.

- Draw the decision tree.
- What would be your advice to the householder if $x = 50$ and $y = 40\%$ and his objective is to maximize expected monetary value (EMV)?
- Formulate the problem of determining the maximum and minimum values of x such that the policy from company B has the highest EMV using linear programming with two variables x and y (i.e. both x and y are now variables, not known constants).

Solution

The decision tree for the problem is shown below.



Below we carry out step 1 of the decision tree solution procedure which (for this example) involves working out the total profit for each of the paths from the initial node to the terminal node.

Step 1

- Path to terminal node 9 - we have no insurance policy but suffer no theft.

Total profit = 0

- Path to terminal node 10 - we have no insurance policy but suffer a theft resulting in a loss of 10% of the contents.

Total cost = $0.1(20000) = 2000$ Total profit = - 2000

Similarly for terminal nodes 11 and 12 total profit = -4000 and -8000 respectively.

- Path to terminal node 13 - we have an insurance policy with company A costing £150 but suffer no theft.

Total cost = 150 Total profit = -150

- Path to terminal node 14 - we have an insurance policy with company A costing £150 but suffer a theft resulting in a loss of $0.1(20000) = £2000$ for which we are *reimbursed* in full by company A. Hence

Total revenue = 2000 Total cost = 2000 + 150 Total profit = -150

It is clear from this calculation that when the reimbursement equals the amount lost the total profit will always be just the cost of the insurance.

This will be the case for terminal nodes 15 and 16 respectively.

Continuing in a similar manner we can arrive at the table below indicating for each branch the total profit involved in that branch from the initial node to the terminal node.

Terminal node	Total profit £
9	0
10	-2000
11	-4000
12	-8000
13	-150
14	-150
15	-150
16	-150
17	-100
18	$-100-x$ ($x \leq 2000$)
19	$-100-x$
20	$-100-x$
21	-75
22	$-75-2000(1-y/100)$
23	$-75-4000(1-y/100)$
24	$-75-8000(1-y/100)$

We can now carry out the second step of the decision tree solution procedure where we work from the right-hand side of the diagram back to the left-hand side.

Step 2

Consider chance node 5 with branches to terminal nodes 10, 11 and 12 emanating from it.

The expected monetary value for this chance node is given by

$$0.5(-2000) + 0.35(-4000) + 0.15(-8000) = -3600$$

Hence the EMV for chance node 1 is given by $0.97(0) + 0.03(-3600) = -108$

Similarly the EMV for chance node 2 is -150.

The EMV for chance node 3 is $0.97(-100) + 0.03[0.5(-100-x) + 0.35(-100-x) + 0.15(-100-x)]$

$$= -97 + 0.03(-100-x) = -100 - 0.03x \quad (x \leq 2000) = -101.5 \text{ since } x = 50$$

The EMV for chance node 4 is

$$0.97(-75) + 0.03[0.5(-75-2000(1-y/100)) + 0.35(-75-4000(1-y/100)) + 0.15(-75-8000(1-y/100))]$$

$$= 0.97(-75) + 0.03[-75-(1-y/100)(3600)] = -75 + 1.08y - 108 = -183 + 1.08y$$

$$= -139.8 \text{ since } y = 40$$

Hence at the initial decision node we have the four alternatives

1. no policy EMV = -108
2. company A policy EMV = -150
3. company B policy EMV = -101.5
4. company C policy EMV = -139.8

Hence the best alternative is the policy from company B leading to an EMV of - £101.5.

Summery

Hundreds of decisions are made every day in the operations activity. Each minor decision determines the company's success or failure. It ranges from simple judgmental to complex analysis which can also involve judgment (past experience and common sense). They involve a way of blending objective and subjective data to arrive at a choice. The use of quantitative methods of analysis adds to the objectivity of such decisions.

*Decision making is the process of selecting or choosing based on some criteria, the best alternative among alternatives. Making appropriate decision is the most **vital** aspect in management .Every one of us takes a number decisions every day. Some are important; some are trivial. Some decisions initiate a set of activities; some put an end to a certain activities. In business environment right decisions at the right times ensure success. This shows the importance of decision making.*

Decisions are made under three basic conditions. Those are: Decision under certainty, Decision under risk and Decision under uncertainty.

Self Test Exercise 4

Part I: Write “True” if the statement is correct and “False” if the statement is incorrect.

1. Decision theory always guarantees optimal solution.
2. Decision theory provides a rational and logical way of making decisions.
3. In evaluating a decision tree, you write the expected payoff at a random outcome node inside the circle representing the node.

Part II: Choose the correct answer & encircle the letter of your choice.

1. Which of the following is not part of decision tree problem specification?
 - A. A list of alternatives.
 - B. A list of possible state of nature.
 - C. Expected value of perfect information.
 - D. Payoff associated with alternative state of nature.
2. If a decision theory problem has 3 decision alternatives and 4 states of nature, the number of payoffs in that problem will be:
 - A. 3
 - B. 4
 - C. 12
 - D. 64
3. In a decision theory problem under complete uncertainty, which one of the following approaches will not be possible?
 - A. Expected monetary value
 - B. Maximin
 - C. Minimax
 - D. Hurwicz

Part III: show all the necessary steps.

Pay off table

Alternatives	Profit if Demand is (Birr)		
	Low (0.1)	Medium (0.5)	High (0.4)
Arrange for sub contract (A_1)	10,000	50,000	50,000
Over time (A_2)	-20,000	60,000	100,000
New facilities (A_3)	-150,000	20,000	200,000

Based on the above pay off table and assuming that there is not probability value of occurrence of each outcome, determine which alternative would be chosen under each of these strategies.

- A. Maximin
- B. Maximax
- C. Laplace
- D. Minimax regret
- E. Expected monetary value

UNIT FIVE: NETWORK MODEL

5. Introduction

Basically, CPM (Critical Path Method) and PERT (Program Evaluation Review Technique) are project management techniques, which have been created out of the need of Western industrial and military establishments to plan, schedule and control complex projects. One of the most popular uses of networks is for project analysis. Such projects as the construction of a building, the development of a drug, or the installation of a computer system can be represented as networks. This network illustrates the way in which the parts of the project are organized, and they can be used to determine the time duration of the projects. Network models consists of a set of circle, or nodes , and lines, which are referred to as either arcs or branches, that connect some nodes to other nodes. Networks are important tools of management science. Not only can networks be used to model a wide variety of problems, they can often be solved more easily than other models of the same problems, and they present models in visual format.

Learning Objectives: at the end of this unit, you will be able to:

- Give a general description of PERT/CPM techniques;
- Understand the concepts of networking;
- *Know the importance of networking for projects.*

5.1. Networking Model

Dear learners, what is networking, CPM and PERT to you? Use the space provided below to write your view.

5.2. Brief History of CPM/PERT

Network models consists of a set of circle, or nodes , and lines, which are referred to as either arcs or branches, that connect some nodes to other nodes. CPM/PERT or Network Analysis

as the technique is sometimes called, developed along two parallel streams, one industrial and the other military. CPM was the discovery of M.R.Walker of E.I.Du Pont de Nemours & Co. and J.E.Kelly of Remington Rand, circa 1957. The computation was designed for the UNIVAC-I computer. The first test was made in 1958, when CPM was applied to the construction of a new chemical plant. In March 1959, the method was applied to maintenance shut-down at the Du Pont works in Louisville, Kentucky. Unproductive time was reduced from 125 to 93 hours.

PERT was devised in 1958 for the POLARIS missile program by the Program Evaluation Branch of the Special Projects office of the U.S.Navy, helped by the Lockheed Missile Systems division and the Consultant firm of Booz-Allen & Hamilton. The calculations were so arranged so that they could be carried out on the IBM Naval Ordnance Research Computer (NORC) at Dahlgren, Virginia.

5.3. Planning, Scheduling & Control

Planning, Scheduling (or organizing) and Control are considered to be basic Managerial functions, and CPM/PERT has been rightfully accorded due importance in the literature on Operations Research and Quantitative Analysis.

Far more than the technical benefits, it was found that PERT/CPM provided a focus around which managers could brain-storm and put their ideas together. It proved to be a great communication medium by which thinkers and planners at one level could communicate their ideas, their doubts and fears to another level. Most important, it became a useful tool for evaluating the performance of individuals and teams. There are many variations of CPM/PERT which have been useful in planning costs, scheduling manpower and machine time. CPM/PERT can answer the following important questions:

How long will the entire project take to be completed? What are the risks involved?

Which are the critical activities or tasks in the project which could delay the entire project if they were not completed on time?

Is the project on schedule, behind schedule or ahead of schedule?

If the project has to be finished earlier than planned, what is the best way to do this at the least cost?

5.4. The Framework for PERT and CPM

Essentially, there are six steps which are common to both the techniques. The procedure is listed below:

- I. Define the Project and all of its significant activities or tasks. The Project (made up of several tasks) should have only a single start activity and a single finish activity.
- II. Develop the relationships among the activities. Decide which activities must precede and which must follow others.
- III. Draw the "Network" connecting all the activities. Each Activity should have unique event numbers. Dummy arrows are used where required to avoid giving the same numbering to two activities.
- IV. Assign time and/or cost estimates to each activity
- V. Compute the longest time path through the network. This is called the critical path.
- VI. Use the Network to help plan, schedule, and monitor and control the project.

The Key Concept used by CPM/PERT is that a small set of activities, which make up the longest path through the activity network control the entire project. If these "critical" activities could be identified and assigned to responsible persons, management resources could be optimally used by concentrating on the few activities which determine the fate of the entire project. Non-critical activities can be replanted, rescheduled and resources for them can be reallocated flexibly, without affecting the whole project.

Five useful questions to ask when preparing an activity network are:

- Is this a Start Activity?
- Is this a Finish Activity?
- What Activity Precedes this?

- What Activity Follows this?
- What Activity is Concurrent with this?

Some activities are serially linked. The second activity can begin only after the first activity is completed. In certain cases, the activities are concurrent, because they are independent of each other and can start simultaneously. This is especially the case in organizations which have supervisory resources so that work can be delegated to various departments which will be responsible for the activities and their completion as planned. When work is delegated like this, the need for constant feedback and co-ordination becomes an important senior management pre-occupation. Network analysis is the general name given to certain specific techniques which can be used for the planning, management and control of projects.

5.5. Use of nodes and arrows

Arrows an arrow leads from tail to head directionally

- Indicate ACTIVITY, a time consuming effort that is required to perform a part of the work.
- Nodes: A node is represented by a circle
 - Indicate EVENT, a point in time where one or more activities start and/or finish activity.
 - A task or a certain amount of work required in the project
 - Requires time to complete
 - Represented by an arrow
- Dummy Activity
 - Indicates only precedence relationships
 - Does not require any time of effort
- Event
 - Signals the beginning or ending of an activity
 - Designates a point in time
 - Represented by a circle (node)

- Network

Shows the sequential relationships among activities using nodes and arrows

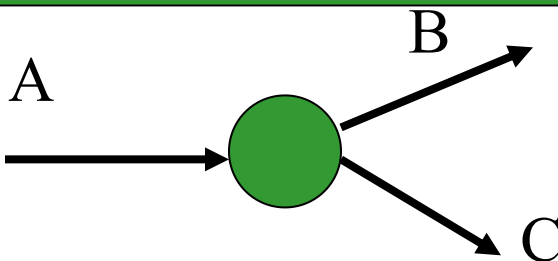
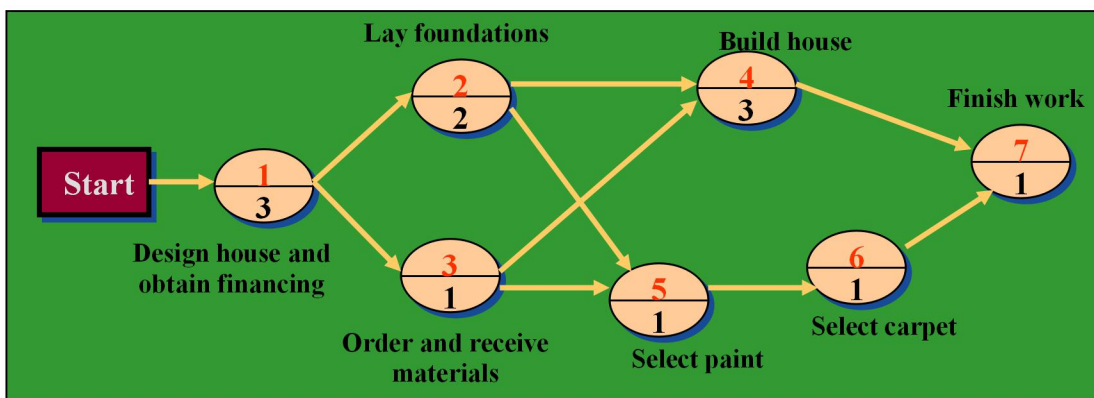
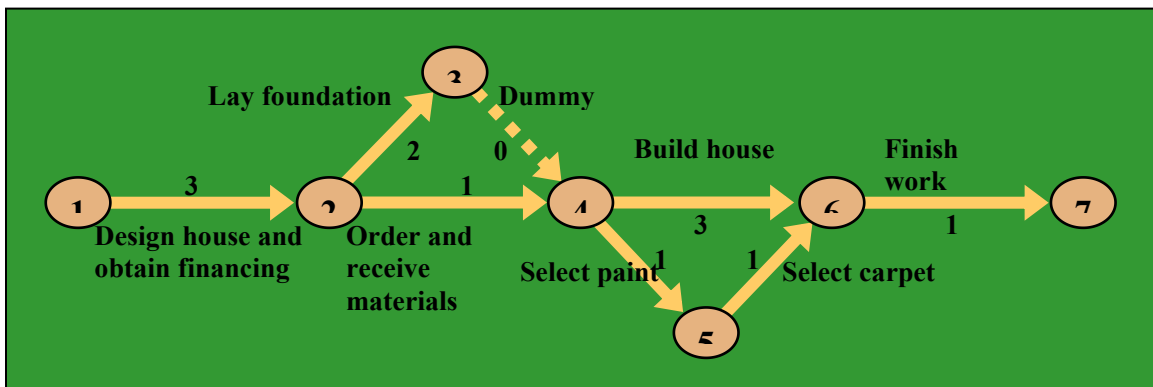
- ♦ Activity-on-node (AON)

Nodes represent activities, and arrows show precedence relationships

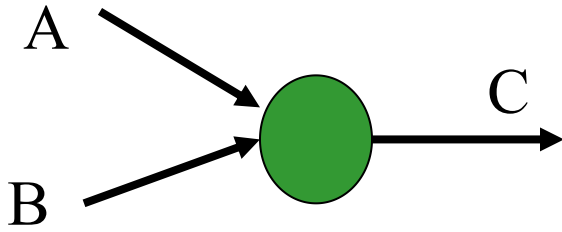
- ♦ Activity-on-arrow (AOA)

Arrows represent activities and nodes are events for points in time

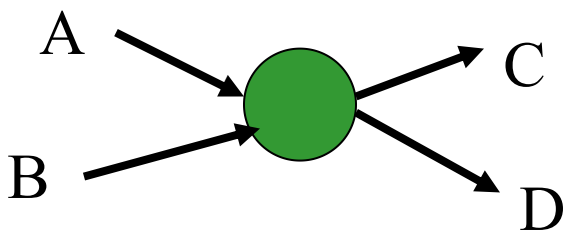
AOA Project Network for House



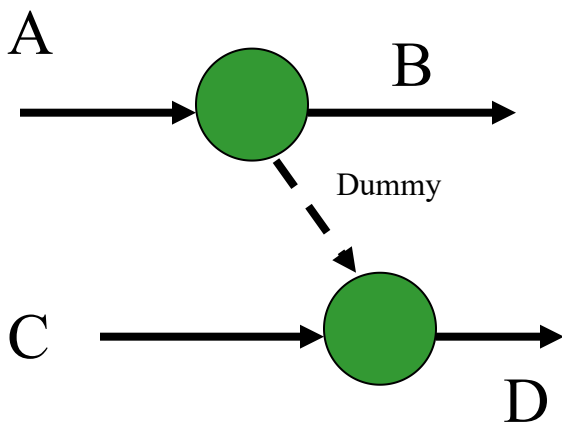
A must finish before either B or C can start



Both A and B must finish before C can start

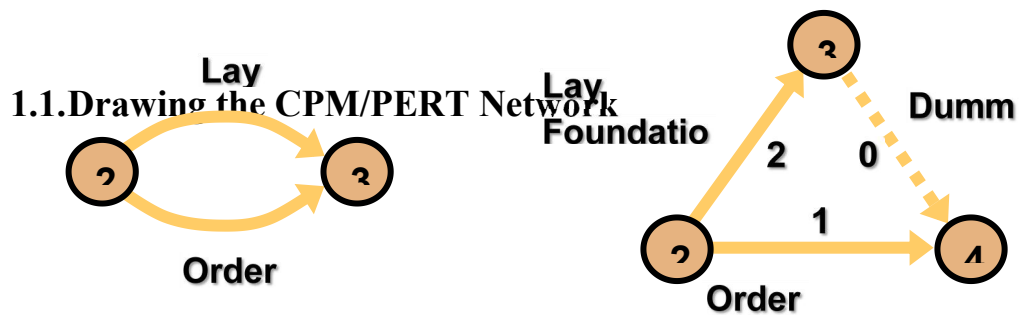


Both A and C must finish before either of B or D can start

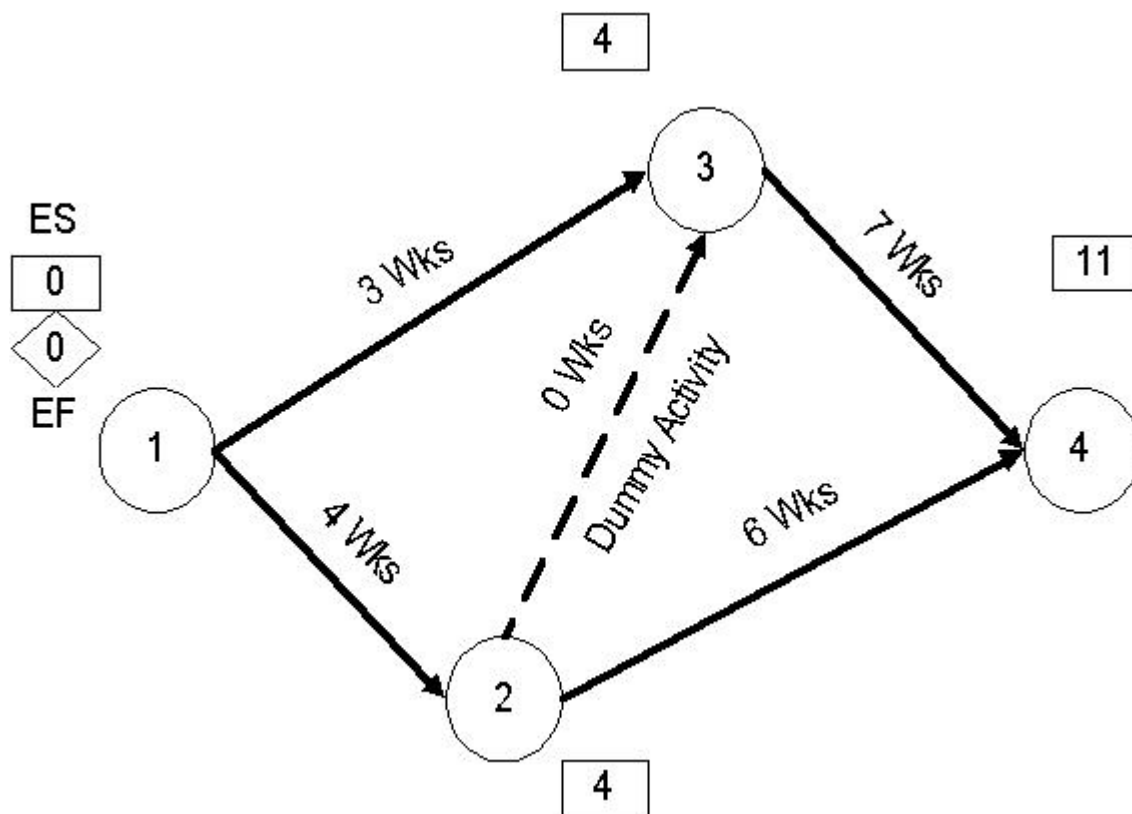


A must finish before B can start both A and C must finish before D can start

Concurrent Activities



Each activity (or sub-project) in a PERT/CPM Network is represented by an arrow symbol. Each activity is preceded and succeeded by an event, represented as a circle and numbered.

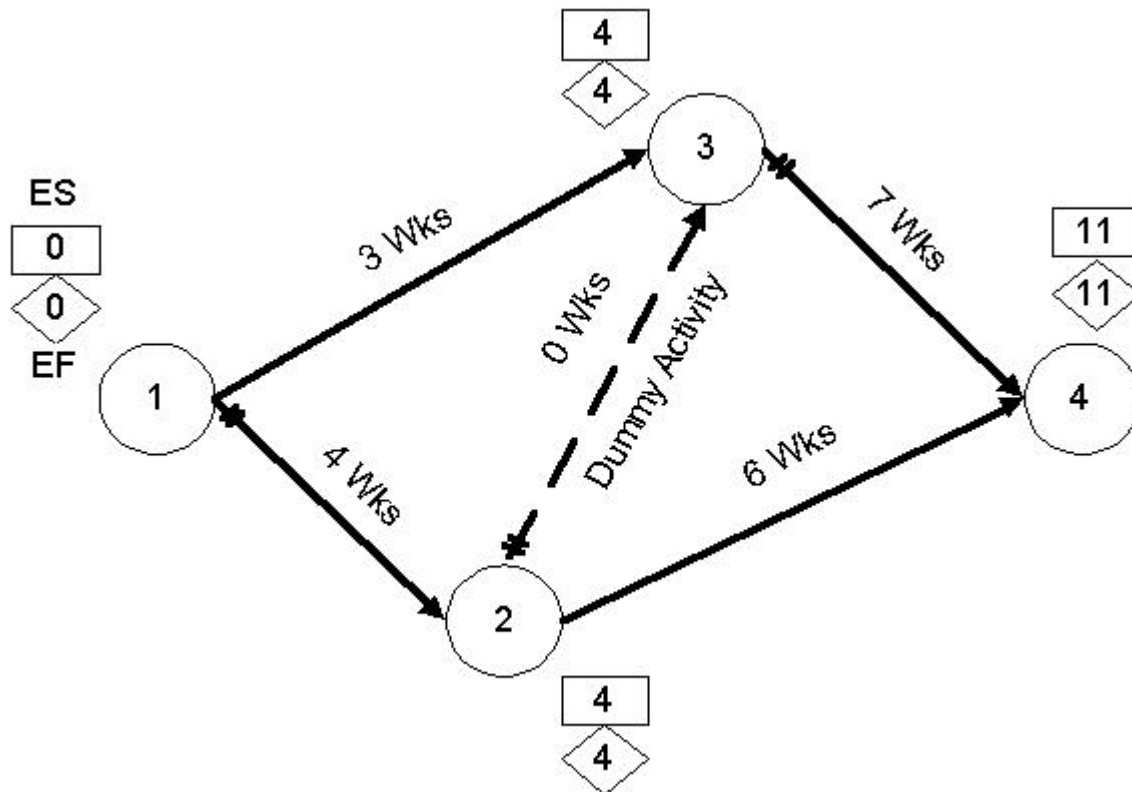


At Event 3, we have to evaluate two predecessor activities - Activity 1-3 and Activity 2-3, both of which are predecessor activities. Activity 1-3 gives us an Earliest Start of 3 weeks at Event 3. However, Activity 2-3 also has to be completed before Event 3 can begin. Along this route, the Earliest Start would be $4+0=4$. The rule is to take the longer (bigger) of the two Earliest Starts. So the Earliest Start at event 3 is 4.

Similarly, at Event 4, we find we have to evaluate two predecessor activities - Activity 2-4 and Activity 3-4. Along Activity 2-4, the Earliest Start at Event 4 would be 10 wks, but along Activity 3-4, the Earliest Start at Event 4 would be 11 wks. Since 11 wks is larger than 10 wks, we select it as the Earliest Start at Event 4. We have now found the longest path through the network. It will take 11 weeks along activities 1-2, 2-3 and 3-4. This is the Critical Path.

5.5.1. The Backward Pass - Latest Finish Time Rule

To make the Backward Pass, we begin at the sink or the final event and work backwards to the first event.



At Event 3 there is only one activity, Activity 3-4 in the backward pass, and we find that the value is $11 - 7 = 4$ weeks. However at Event 2 we have to evaluate 2 activities, 2-3 and 2-4. We find that the backward pass through 2-4 gives us a value of $11 - 6 = 5$ while 2-3 gives us $4 - 0 = 4$. We take the **smaller value** of 4 on the backward pass.

5.5.2. Tabulation & Analysis of Activities

We are now ready to tabulate the various events and calculate the Earliest and Latest Start and Finish times. We are also now ready to compute the SLACK or TOTAL FLOAT, which is defined as the difference between the Latest Start and Earliest Start.

Event	Duration(Weeks)	Earliest Start	Earliest Finish	Latest Start	Latest Finish	Total Float
1-2	4	0	4	0	4	0
2-3	0	4	4	4	4	0
3-4	7	4	11	4	11	0
1-3	3	0	3	1	4	1
2-4	6	4	10	5	11	1

- The Earliest Start is the value in the rectangle near the tail of each activity
- The Earliest Finish is = Earliest Start + Duration
- The Latest Finish is the value in the diamond at the head of each activity
- The Latest Start is = Latest Finish - Duration

There are two important types of Float or Slack. These are Total Float and Free Float.

TOTAL FLOAT is the spare time available when all preceding activities occur at the **earliest** possible times and all succeeding activities occur at the **latest** possible times.

- **Total Float = Latest Start - Earliest Start**

Activities with zero Total float are on the Critical Path

FREE FLOAT is the spare time available when all preceding activities occur at the **earliest** possible times and all succeeding activities occur at the **earliest** possible times.

When an activity has zero Total float, free float will also be zero.

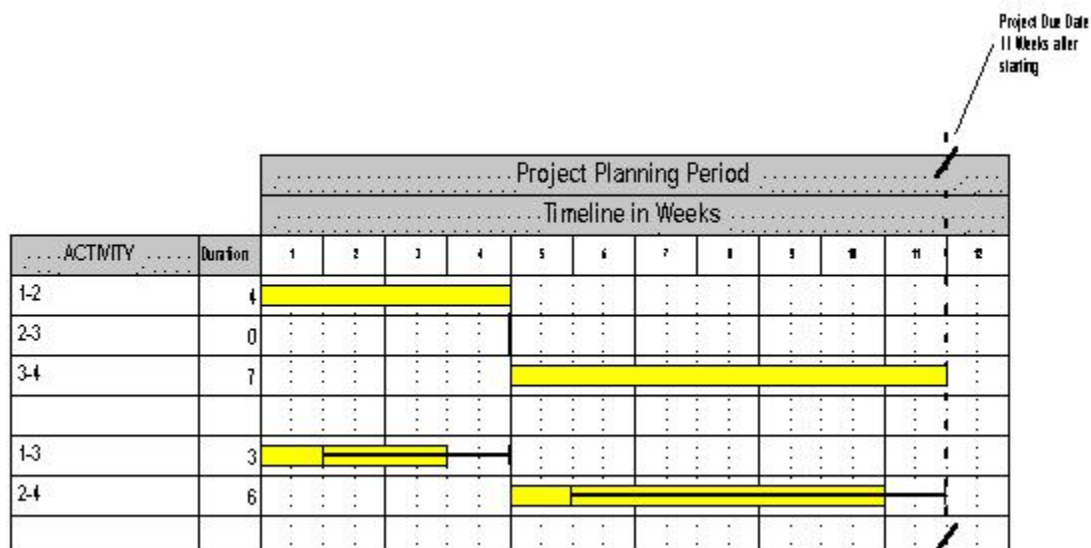
There are various other types of float (Independent, Early Free, Early Interfering, Late Free, Late Interfering), and float can also be negative. We shall not go into these situations at present for the sake of simplicity and be concerned only with Total Float for the time being.

Having computed the various parameters of each activity, we are now ready to go into the scheduling phase, using a type of bar chart known as the Gantt chart.

There are various other types of float (Independent, Early Free, Early Interfering, Late Free, Late Interfering), and float can also be negative. We shall not go into these situations at present for the sake of simplicity and be concerned only with Total Float for the time being. Having computed the various parameters of each activity, we are now ready to go into the scheduling phase, using a type of bar chart known as the Gantt Chart.

5.6. Scheduling of Activities Using a Gantt chart

Once the activities are laid out along a Gantt chart (Please see chart below), the concepts of Earliest Start & Finish, Latest Start & Finish and Float will become very obvious.



Activities 1-3 and 2-4 have total float of 1 week each, represented by the solid timeline which begins at the latest start and ends at the latest finish. The difference is the float, which gives us the flexibility to schedule the activity.

For example, we might send the staff on leave during that one week or give them some other work to do. Or we may choose to start the activity slightly later than planned, knowing that we have a week is float in hand. We might even break the activity in the middle (if this is permitted) for a week and divert the staff for some other work, or declare a National or Festival holiday as required under the National and Festival Holidays Act.

These are some of the examples of the use of float to schedule an activity. Once all the activities that can be scheduled are scheduled to the convenience of the project, normally reflecting resource optimization measures, we can say that the project has been scheduled.

Benefits of CPM/PERT

- Useful at many stages of project management
- Mathematically simple
- Give critical path and slack time
- Provide project documentation
- Useful in monitoring costs

Limitations to CPM/PERT

- Clearly defined, independent and stable activities
- Specified precedence relationships
- Over emphasis on critical paths
- Deterministic CPM model
- Activity time estimates are subjective and depend on judgment
- PERT assumes a beta distribution for these time estimates, but the actual distribution may be different
- PERT consistently underestimates the expected project completion time due to alternate paths becoming critical

To overcome the limitation, Monte Carlo simulations can be performed on the network to eliminate the optimistic bias

Example

A Social Project manager is faced with a project with the following activities:

Activity-id	Activity – Description	Duration
1-2	Social Work Team to live in Village	5 Weeks
1-3	Social Research Team to do survey	12 Weeks
3-4	Analyze results of survey	5 Weeks
2-4	Establish Mother & Child Health Program	14 Weeks
3-5	Establish Rural Credit Program	15 Weeks
4-5	Carry out Immunization of Under Fives	4 Weeks

- Draw the arrow diagram, using the helpful numbering of the activities, which suggests the following logic:
- Unless the Social Work team lives in the village, the Mother and Child Health Program cannot be started due to ignorance and superstition of the villagers
- The Analysis of the survey can obviously be done only after the survey is complete.
- Until rural survey is done, the Rural Credit Program cannot be started
- Unless Mother and Child Program is established, the Immunization of Under Fives cannot be started
- - Calculate the Earliest and Latest Event Times
- - Tabulate and Analyze the Activities
- - Schedule the Project Using a Gantt Chart

5.7. The PERT (Probabilistic) Approach

So far we have talked about projects, where there is high certainty about the outcomes of activities. In other words, the cause-effect logic is well known. This is particularly the case in engineering projects.

However, in Research & Development projects, or in Social Projects which are defined as "Process Projects", where learning is an important outcome, the cause-effect relationship is not so well established.

In such situations, the PERT approach is useful, because it can accommodate the variation in event completion times, based on an expert's or an expert committee's estimates.

For each activity, three time estimates are taken

PERT duration estimates:

- Optimistic duration (t_O): This is the time a task would take if no unexpected risks happen during the execution of a task and everything goes perfectly smooth. (This is what an inexperienced manager believes!)
- Most likely duration (t_M): Most realistic time estimate to complete the task. This includes estimating and planning risk contingencies that are likely to be put into use during task execution. Seasoned managers have an uncanny way of estimating very close to the actual time using historical data from prior estimation errors.
- Pessimistic duration (t_P): Duration a task would take if everything goes wrong. It assumes all possible risks happening with the project.

The Duration of an activity is calculated using the following formula:

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

Where t_e is the Expected time, t_o is the Optimistic time, t_m is the most probable activity time and t_p is the Pessimistic time.

It is not necessary to go into the theory behind the formula. It is enough to know that the weights are based on an approximation of the Beta distribution.

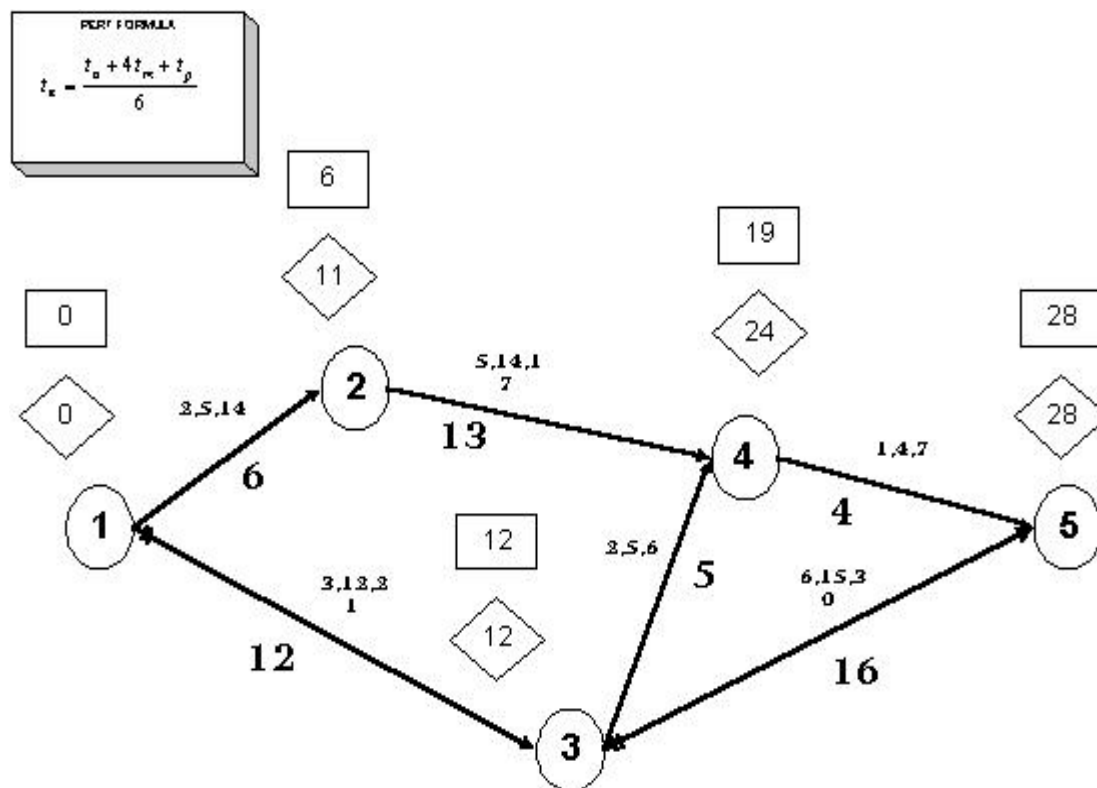
The Standard Deviation, which is a good measure of the variability of each activity is calculated by the rather simplified formula:

$$s_1 = \frac{t_p - t_o}{6}$$

The Variance is the Square of the Standard Deviation.

5.7. 1. PERT Calculations for the Social Project

In our Social Project, the Project Manager is now not so certain that each activity will be completed on the basis of the single estimate he gave. There are many assumptions involved in each estimate, and these assumptions are illustrated in the three-time estimate he would prefer to give to each activity.



In Activity 1-3, the time estimates are 3, 12 and 21. Using our PERT formula, we get:

$$t_e = \frac{3 + (4 \times 12) + 21}{6} = \frac{72}{6} = 12$$

The Standard Deviation (s.d.) for this activity is also calculated using the PERT formula

$$s_1 = \frac{(21-3)}{6} = \frac{18}{6} = 3$$

We calculate the PERT event times and other details as below for each activity:

Event	t _o	t _m	t _p	t _e	ES	EF	LS	LF	TF	s.d.	Var.
1-3	3	12	21	12	0	12	0	12	0	3	9
3-5	6	15	30	16	12	28	12	28	0	4	16
1-2	2	5	14	6	0	6	5	11	5	2	4
2-4	5	14	17	13	6	19	11	24	5	2	4
3-4	2	5	8	5	12	17	19	24	7	1	1
4-5	1	4	7	4	19	23	24	28	5	1	1

5.8. Estimating Risk

Having calculated the s.d. and the Variance, we are ready to do some risk analysis. Before that we should be aware of two of the most important assumptions made by PERT.

- The Beta distribution is appropriate for calculation of activity durations.

- Activities are independent, and the time required to complete one activity has no bearing on the completion times of its successor activities in the network. The validity of this assumption is questionable when we consider that in practice, many activities have dependencies.

5.8.1. Expected Length of a Project

PERT assumes that the expected length of a project (or a sequence of independent activities) is simply the sum of their separate expected lengths.

Thus the summation of all the t_e 's along the critical path gives us the length of the project.

Similarly the variance of a sum of independent activity times is equal to the sum of their individual variances.

In our example, the sum of the variance of the activity times along the critical path, VT is found to be equal to $(9+16) = 25$.

The square root VT gives us the standard deviation of the project length. Thus, $ST = \sqrt{25} = 5$. The higher the standard deviation, the greater the uncertainty that the project will be completed on the due date

Although the t_e 's are randomly distributed, the average or expected project length T_e approximately follows a Normal Distribution.

Since we have a lot of information about a Normal Distribution, we can make several statistically significant conclusions from these calculations.

A random variable drawn from a Normal Distribution has 0.68 probability of falling within one standard deviation of the distribution average. Therefore, there is a 68% chance that the actual project duration will be within one standard deviation, ST of the estimated average length of the project, t_e .

In our case, the $t_e = (12+16) = 28$ weeks and the $ST = 5$ weeks. Assuming t_e to be normally distributed, we can state that there is a probability of 0.68 that the project will be completed within 28 - 5 weeks, which is to say, between 23 and 33 weeks.

Since it is known that just over 95% (.954) of the area under a Normal Distribution falls within two standard deviations, we can state that the probability that the project will be completed within 28 - 10 is very high at 0.95.

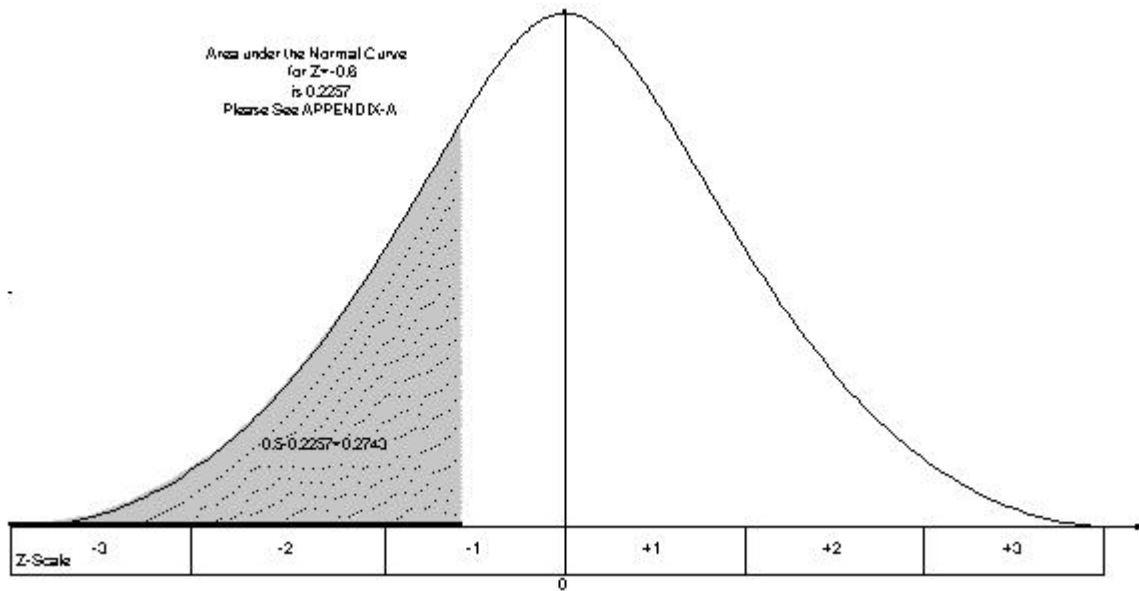
5.8.2. Probability of Project Completion by Due Date

Now, although the project is estimated to be completed within 28 weeks ($t_e=28$) our Project Director would like to know what is the probability that the project might be completed within 25 weeks (i.e. Due Date or $D=25$).

For this calculation, we use the formula for calculating Z , the number of standard deviations that D is away from t_e .

$$Z = \frac{D - t_e}{S_t} = \frac{25 - 28}{5} = \frac{-3}{5} = -0.6$$

By looking at the following extract from a standard normal table, we see that the probability associated with a Z of -0.6 is 0.274. This means that the chance of the project being completed within 25 weeks, instead of the expected 28 weeks is about 2 out of 7. Not very encouraging.

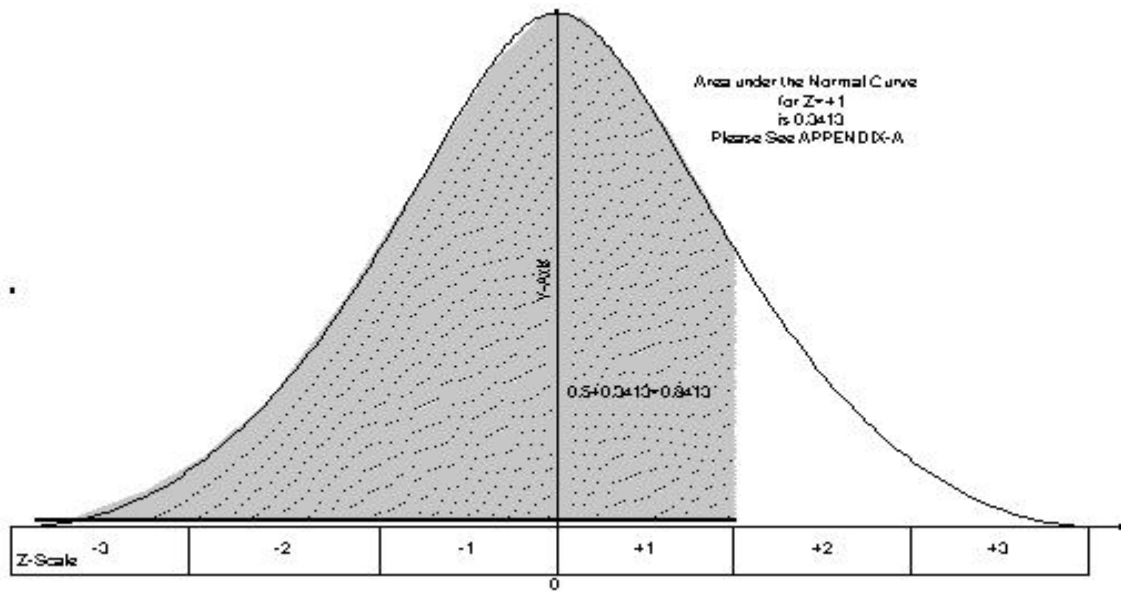


On the other hand, the probability that the project will be completed within 33 weeks is calculated as follows:

$$Z = \frac{D - t_e}{S_t} = \frac{33 - 28}{5} = \frac{5}{5} = 1$$

The probability associated with $Z = +1$ is 0.84134.

This is a strong probability, and indicates that the odds are 16 to 3 that the project will be completed by the due date.



If the probability of an event is p , the odds for its occurrence are a to b , where:

$$\frac{a}{b} = \frac{p}{1-p} = \frac{0.84134}{0.15866} \approx \frac{16}{3}$$

Problems on Network

Problem 1

House Building Activities are given as shown on the table below: You are required to

1. Draw the project network, determine the critical path and setup LP.

2. Determine the total float and free float for each activity.

Activity	Description	Immediate Predecessors	Duration (days)
A	Build Foundation	–	5
B	Build walls, ceilings	A	8
C	Build Roof	B	10
D	Do wiring	B	5
E	Put in windows	B	4
F	Put on siding	E	6
G	Paint house	C,F	3

Problem 2;

Illustration of network analysis of a minor redesign of a product and its associated packaging

The key question is: How long will it take to complete this project?

Activity number		Completion time (weeks)
1	Redesign product	6
2	Redesign packaging	2
3	Order and receive components for redesigned product	3
4	Order and receive material for redesigned packaging	2
5	Assemble products	4
6	Make up packaging	1
7	Package redesigned product	1
8	Test market redesigned product	6
9	Revise redesigned product	3
10	Revise redesigned packaging	1
11	Present results to the Board	1

For clarity, this list is kept to a minimum by specifying only immediate relationships, which is a relationship involving activities that "occur near to each other in time".

Activity number		Activity number
1	must be finished before	3
2		4
3		5
4		6
5, 6		7
7		8
8		9
8		10
9, 10		11

Summery

Networks are important tools of management science. Not only can networks be used to model a wide variety of problems, they can often be solved more easily than other models of the same problems, and they present models in visual format.

*One of the most popular uses of networks is for project analysis. Such projects as the construction of a building, the development of a drug, or the installation of a computer system can be represented as networks. These networks illustrate the way in which the parts of the project are organized, and they can be used to determine the time duration of the projects. Network models consists of a set of circle, or nodes , and lines, which are referred to as either arcs or branches, that connect some nodes to other nodes. The two closely related operations research techniques, **PERT** (program evaluation and review technique) and **CPM** (critical path method), are available to assist the project manager in carrying out these responsibilities. These techniques make heavy use of networks to help plan and display the coordination of all the activities. They also normally use a software package to deal with all the data needed to develop schedule information and then to monitor the progress of the project. PERT and CPM has been used for a variety of projects, including the following types: Construction of a new plant, Research and development of a new product, NASA space exploration projects etc.*

Self Test Exercise 5

Part I: Choose the correct answer & encircle the letter of your choice.

1. PERT and CPM can be used for all the following except one;

- A. Construction of a new plant
- B. Research and development of a new product
- C. Building a ship
- D. Government-sponsored projects for developing a new weapons system
- E. Relocation of a major facility
- F. None

2. _____ is useful for analyzing project scheduling problems in which the completion time of the different activities, and therefore the whole project, is not certain.

- A. PERT
- B. CPM
- C. Both
- D. None

3. _____ is most appropriately used in projects in which the activity durations are known with certainty.

- A. All projects
- B. PERT
- C. CPM
- D. None

Part II: show all the necessary steps

1. Draw the project network, determine the critical path and setup LP.
2. Determine the total float and free float for each activity.

Activity	Description	Immediate Predecessors	Duration (days)
A	Build Foundation	–	5
B	Build walls, ceilings	A	8
C	Build Roof	B	10
D	Do wiring	B	5
E	Put in windows	B	4
F	Put on siding	E	6
G	Paint house	C,F	3

UNIT SIX: GAME THEORY

6. Introduction

Dear Learner, in this unit you are going to deal with game theory. Game theory is a tool that can help explain and address social problems. Since games often reflect or share characteristics with real situations especially competitive or cooperative situations -- they can suggest strategies for dealing with such circumstances. Just as we may be able to understand the strategy of players in a particular game, we may also be able to predict how people, political factions, or states will behave in a given situation.

Just as people generally try to win games, people also try to "win" or achieve their interests or goals in competitive situations. However, both in games and in the real world, we generally follow a set of rules to do this. Some games, like some real situations are "winner-take-all." These games are by their nature very competitive, as only one person can win. (Chess would be an example of such a game.) Other games, however, require cooperation to win. Many of the newer video games, for example, require cooperative strategies among multiple players in order for any single player to advance. In the real world, even during times of hostility, rivals generally have common interests and must cooperate to some degree. Even during the Cold War, despite an intense East-West standoff, Moscow and Washington cooperated to achieve their common goal of averting a nuclear war.

Learning Objectives: at the end of this unit, you will be able to:

- Know what is the meaning of game theory;
- Understand the importance of game theory;
- Understand two person zero sum game;
- Identify the best strategy for players in a game theory.

6.1. Meaning of Game Theory

Dear learners, how are you going to define game theory? Use the space below to express your feelings.

6.1.1. What is Game Theory?

Game theory provides analytical tools for examining strategic interactions among two or more participants. By using simple, often numerical models to study complex social relations, game theory can illustrate the potential for, and risks associated with, cooperative behavior among distrustful participants. Though less familiar than typical board or video games the lessons from these more abstract or hypothetical games are applicable to a wider array of social situations.

Games used to simulate real-life situations typically include five elements:

1. *players*, or decision makers;
2. *strategies* available to each player;
3. *rules* governing players' behavior;
4. *outcomes*, each of which is a result of particular choices made by players at a given point in the game; and
5. *payoffs* accrued by each player as a result of each possible outcome.[2]

These games assume that each player will pursue the strategies that help him or her to achieve the most profitable outcome in every situation. Real life is full of situations in which people -- intentionally or unintentionally -- pursue their own interests at the expense of others, leading to conflict or competition. Games used to illustrate these relationships often place the interests of two players in direct opposition: the greater the payoff for one player, the less for the other. In order to achieve a mutually productive outcome, the players must coordinate their strategies, because if each player pursues his or her greatest potential payoffs, the shared outcome is unproductive. This concept is illustrated below, using the *Prisoner's Dilemma Game*.

This and other games illustrate the potential for cooperation to produce mutually beneficial outcomes. However, they also highlight the difficulties of obtaining cooperation among distrustful participants, because each player is tempted to pursue his or her individual interests. Cooperation requires that both players compromise, and forego their individual maximum payoffs. Yet, in compromising, each player risks complete loss if the opponent decides to seek his or her own maximum payoff. Rather than risking total loss, players tend to prefer the less productive outcome.

6.2. Why is Game Theory Useful?

These models can provide insight into the strategic options and likely outcomes available to participants in particular situations. From this insight, decision-makers can better assess the potential effects of their actions, and can make decisions that will more likely produce the desired goals and avoid conflict.

For example, deterrence theory has guided U.S. defense strategy since the end of World War II. It assumes that a credible and significant threat of retaliation can curb an aggressor's behavior; if an individual believes that aggressive behavior may trigger an unacceptable and violent response from others, he or she is less likely to behave aggressively. The threat of reprisal does not directly reduce the probability of violence; instead, the perceived benefit of aggressive behavior decreases, in the face of probable retaliation. If two individuals recognize that their best interests lie in avoiding each other's retaliation, neither is likely to initiate hostilities. This was the guiding principle behind U.S.-Soviet relations during much of the Cold War. The concept of mutual deterrence paved the way for arms-control measures and further cooperation. By highlighting strategic choices and potential collective outcomes, game theory helped illustrate how a potentially destructive relationship could be framed, managed, and transformed to provide mutual benefits, including avoidance of an uncontrolled arms race and nuclear war.

6.3. The Prisoner's Dilemma

The Prisoner's Dilemma, illustrated in *Figure 5.1*, is one of the best-known models in game theory. It illustrates the paradoxical nature of interaction between mutually suspicious participants with opposing interests.

Figure 5.1. Possible outcomes for the Prisoner's Dilemma. The number in the upper triangle of each pair indicates the payoff for Player B; the lower triangle, Player A. Higher numbers represent greater payoff for the individual. The corresponding order of preference for these options decreases from 4 (most preferred) to 1 (least preferred).

		Player A	
		Cooperate	Defect
Player B	Cooperate	3 / 3	1 / 4
	Defect	4 / 1	2 / 2

In this hypothetical situation, two accomplices to a crime are imprisoned, and they forge a pact to not betray one another and not confess to the crime. The severity of the punishment that each receives is determined not only by his or her behavior, but also by the behavior of his or her accomplice. The two prisoners are separated and cannot communicate with each other. Each is told that there are four possible outcomes:

1. If one confesses to the crime and turns in the accomplice (*defecting* from a pact with the accomplice), his sentence will be reduced.
2. If one confesses while the accomplice does not (i.e. the accomplice *cooperates* with the pact to not betray each other), the first can strike a deal with the police, and will be set free. But the information he provides will be used to incriminate his accomplice, who will receive the maximum sentence.
3. If both prisoners confess to the crime (i.e. both *defect* from their pact), then each receives a reduced sentence, but neither is set free.

4. If neither confesses to the crime (i.e. they *cooperate*), then each receives the minimum sentence because of the lack of evidence. This option may not be as attractive to either individual as the option of striking a deal with the police and being set free at the expense of one's partner. Since the prisoners cannot communicate with each other, the question of whether to "trust" the other not to confess is the critical aspect of this game.

Although this is a simple model, its lessons can be used to examine more complex strategic interactions, such as arms races. If two antagonistic countries uncontrollably build up their armaments, they increase the potential for mutual loss and destruction. For each country, the value of arming itself is decreased because the costs of doing so -- financial costs, heightened security tensions, greater mutual destructive capabilities, etc. -- provide few advantages over the opponent, resulting in an unproductive outcome (2 to 2 in Figure 5.1). Each country has a choice: cooperate to control arms development, with the goal of achieving mutual benefits, or defect from the pact, and develop armaments.

The dilemma stems from the realization that if one side arms itself (defects) and the other does not (cooperates), the participant who develops armaments will be considered stronger and will win the game (the 4 to 1 outcome). If both cooperate, the best possible outcome is a tie (3 to 3). This is better than the payoff from mutual defection and an arms race (2 to 2), but it is not as attractive as winning, and so the temptation to out-arm one's opponent is always present. The fear that one's opponent will give in to such temptations often drives both players to arm; not doing so risks total loss, and the benefits of not arming can only be realized if one's opponent overcomes his or her temptation to win. Such trust is often lacking in the international environment.

The U.S.-Soviet relationship was a good example of this dynamic. For a long time, the two countries did not trust each other at all. Each armed itself to the hilt, fearing that the other one was doing so, and not wanting to risk being vulnerable. Yet the cost of the arms race was so high that it eventually bankrupted the Soviet Union. Had the Soviets been willing to trust the U.S. more, and vice versa, much of the arms race could have been prevented, at tremendous financial and security savings for both nations, and indeed, the rest of the world.

The lessons initially drawn from the Prisoner's Dilemma can be discouraging. The game illustrates a zero-sum situation, in which one person must lose in order for the other to win. To keep from losing, each player is motivated to pursue a "winning" strategy. The collective result is unproductive, at best, and destructive, at worst.

6.4. A More Realistic Model: Extensions of the Prisoner's Dilemma

Few social situations can be modeled accurately by a single interaction. Rather, most situations result from a series of interactions over a long period of time. An extended version of the Prisoner's Dilemma scenario includes repeated interaction, which increases the probability of cooperative behavior.

The logic of this version of Prisoner's Dilemma suggests that a player's strategy (defect or cooperate) depends on his or her experience in previous interactions, and that that strategy will also affect the future behavior of one's opponent. The result is a relationship of mutual reciprocity; a player is likely to cooperate if his or her opponent previously demonstrated willingness to cooperate, and is unlikely to cooperate if the opponent previously did not. The knowledge that the game will be played again leads players to consider the consequences of their actions; one's opponent may retaliate or be unwilling to cooperate in the future, if one's strategy always seeks maximum payoffs at the expense of the other player.

In a computer-simulated experiment, Robert Axelrod demonstrated that the "winning" strategy in a repeated prisoner's dilemma is one that he terms "tit-for-tat."^[3] This strategy calls for cooperation on the first move, and in each subsequent move, one chooses the behavior demonstrated by one's opponent in the previous round. Still, there is no "right" or best solution to the paradox presented by Prisoner's Dilemma. One lost round in a two-player game can be devastating for a player, and the temptation to defect always exists.

6.5. Zero-Sum Games

By the time Tucker invented the Prisoners' Dilemma, Game Theory was already a going concern. But most of the earlier work had focused on a special class of games: zero-sum games.

In his earliest work, von Neumann made a striking discovery. He found that if poker players maximize their rewards, they do so by bluffing; and, more generally, that in many games it pays to be unpredictable. This was not qualitatively new, of course -- baseball pitchers were throwing change-up pitches before von Neumann wrote about mixed strategies. But von Neumann's discovery was a bit more than just that. He discovered a unique and unequivocal answer to the question "how can I maximize my rewards in this sort of game?" without any markets, prices, property rights, or other institutions in the picture. It was a very major extension of the concept of absolute rationality in neoclassical economics. But von Neumann had bought his discovery at a price. The price was a strong simplifying assumption: von Neumann's discovery applied only to zero-sum games.

For example, consider the children's game of "Matching Pennies." In this game, the two players agree that one will be "even" and the other will be "odd." Each one then shows a penny. The pennies are shown simultaneously, and each player may show either a head or a tail. If both show the same side, then "even" wins the penny from "odd;" or if they show different sides, "odd" wins the penny from "even". Here is the payoff table for the game.

Table 6-1

		Odd	
		Head	Tail
Even	Head	1,-1	-1,1
	Tail	-1,1	1,-1

If we add up the payoffs in each cell, we find $1-1=0$. This is a "zero-sum game."

6.5.1. Definition: Zero-Sum game:

if we add up the wins and losses in a game, treating losses as negatives, and we find that the sum is zero for each set of strategies chosen, and then the game is a "zero-sum game."

In less formal terms, a zero-sum game is a game in which one player's winnings equal the other player's losses. Do notice that the definition requires a zero sum for every set of strategies. If there is even one strategy set for which the sum differs from zero, then the game is not zero sum.

Here is another example of a zero-sum game. It is a very simplified model of price competition. Like Augustin Cournot (writing in the 1840's) we will think of two companies that sell mineral water. Each company has a fixed cost of \$5000 per period, regardless whether they sell anything or not. We will call the companies Perrier and Apollinaris, just to take two names at random.

The two companies are competing for the same market and each firm must choose a high price (\$2 per bottle) or a low price (\$1 per bottle). Here are the rules of the game:

- 1) At a price of \$2, 5000 bottles can be sold for total revenue of \$10000.
- 2) At a price of \$1, 10000 bottles can be sold for total revenue of \$10000.
- 3) If both companies charge the same price, they split the sales evenly between them.
- 4) If one company charges a higher price, the company with the lower price sells the whole amount and the company with the higher price sells nothing.
- 5) Payoffs are profits -- revenue minus the \$5000 fixed cost.

Here is the payoff table for these two companies

Table 6-2

		Perrier	
		Price=\$1	Price=\$2
Appollinaris	Price=\$1	0,0	5000, -5000
	Price=\$2	-5000, 5000	0,0

(Verify for yourself that this is a zero-sum game.) For two-person zero-sum games, there is a clear concept of a solution. The solution to the game is the maxi-min criterion -- that is, each player chooses the strategy that maximizes her minimum payoff. In this game, Appollinaris' minimum payoff at a price of \$1 is zero, and at a price of \$2 it is -5000, so the \$1 price maximizes the minimum payoff. The same reasoning applies to Perrier, so both will choose the \$1 price. Here is the reasoning behind the maxi-min solution: Appollinaris knows that whatever she loses Perrier gains; so whatever strategy she chooses, Perrier will choose the strategy that gives the minimum payoff for that row. Again, Perrier reasons conversely.

Solution: Maxi-min criterion For a two-person, zero sum game it is rational for each player to choose the strategy that maximizes the minimum payoff, and the pair of strategies and payoffs such that each player maximizes her minimum payoff is the "solution to the game."

6.5.2. Mixed Strategies

Now let's look back at the game of matching pennies. It appears that this game does not have a unique solution. The minimum payoff for each of the two strategies is the same: -1. But this is not the whole story. This game can have more than two strategies. In addition to the two obvious strategies, head and tail, a player can "randomize" her strategy by offering either a head or a tail, at random, with specific probabilities. Such a randomized strategy is called a "mixed strategy." The obvious two strategies, heads and tails, are called "pure strategies." There are infinitely many mixed strategies corresponding to the infinitely many ways probabilities can be assigned to the two pure strategies.

Definition Mixed strategy If a player in a game chooses among two or more strategies at random according to specific probabilities, this choice is called a "mixed strategy."

The game of matching pennies has a solution in mixed strategies, and it is to offer heads or tails at random with probabilities 0.5 each way. Here is the reasoning: if odd offers heads with any probability greater than 0.5, then even can have better than even odds of winning by offering heads with probability 1. On the other hand, if odd offers heads with any probability less than 0.5, then even can have better than even odds of winning by offering tails with probability 1. The only way odd can get even odds of winning is to choose a randomized strategy with probability 0.5, and there is no way odd can get better than even odds. The 0.5 probability maximizes the minimum payoff over all pure *or mixed* strategies. And even can reason the same way (reversing heads and tails) and come to the same conclusion, so both players choose 0.5.

6.5.3. Von Neumann's Discovery

We can now say more exactly what von Neumann's discovery was. Von Neumann showed that every two-person zero sum game had a maxi-min solution, in mixed if not in pure strategies. This was an important insight, but it probably seemed more important at the time than it does now. In limiting his analysis to two-person zero sum games, von Neumann had made a strong simplifying assumption. Von Neumann was a mathematician, and he had used the mathematician's approach: take a simple example, solve it, and then try to extend the solution to the more complex cases. But the mathematician's approach did not work as well in game theory as it does in some other cases. Von Neumann's solution applies unequivocally only to "games" that share this zero-sum property. Because of this assumption, von Neumann's brilliant solution was and is only applicable to a small proportion of all "games," serious and non serious. Arms races, for example, are not zero-sum games. Both participants can and often do lose. The Prisoners' Dilemma, as we have already noticed, is not a zero-sum game, and that is the source of a major part of its interest. Economic competition is not a zero-sum game. It is often possible for most players to win, and in principle, economics is a win-win game. Environmental pollution and the overexploitation of resources, again, tend to be lose-lose games: it is hard to find a winner in the destruction of most of the world's ocean

fisheries in the past generation. Thus, von Neumann's solution does not -- without further work -- apply to these serious interactions.

The serious interactions are instances of "non constant sum games," since the winnings and losses may add up differently depending on the strategies the participants choose. It is possible, for example, for rival nations to choose mutual disarmament, save the cost of weapons, and both are better off as a result -- so the sum of the winnings is greater in that case. In economic competition, increasing division of labor, specialization, investment, and improved coordination can increase "the size of the pie," leading to "that universal opulence which extends itself to the lowest ranks of the people," in the words of Adam Smith. In cases of environmental pollution, the benefits to each individual from the polluting activity is so swamped by others' losses from polluting activity that all can lose -- as we have often observed.

Poker and baseball are zero-sum games. It begins to seem that the only zero-sum games are literal games that human beings have invented -- and made them zero-sum -- for our own amusement. "Games" that are in some sense natural are non-constant sum games. And even poker and baseball are somewhat unclear cases. A "friendly" poker game is zero-sum, but in a casino game, the house takes a proportion of the pot, so the sum of the winnings is less the more the players bet. And even in the friendly game, we are considering only the money payoffs -- not the thrill of gambling and the pleasure of the social event, without which presumably the players would not play. When we take those rewards into account, even gambling games are not really zero-sum.

Von Neumann and Morgenstern hoped to extend their analysis to non-constant sum games with many participants, and they proposed an analysis of these games. However, the problem was much more difficult, and while a number of solutions have been proposed, there is no one generally accepted mathematical solution of non constant sum games. To put it a little differently, there seems to be no clear answer to the question, "Just what is rational in a non-constant sum game?" The well-defined rational policy in neoclassical economics -- maximization of reward -- is extended to zero-sum games but not to the more realistic category of non-constant sum games.

6.5.4. Investment Decisions: Optimal Portfolio Selections

Consider the following investment problem discussed in the Decision Analysis site. The problem is to decide what action or a combination of actions to take among three possible courses of action with the given rates of return as shown in the body of the following table.

		States of Nature (Events)			
		Growth	Medium G	No Change	Low
		G	MG	N	L
Actions	Bonds	12%	8	7	3
	Stocks	15	9	5	-2
	Deposit	7	7	7	7

In decision analysis, the decision-maker has to select at least and at most one option from all possible options. This certainly limits its scope and its applications. You have already learned both decision analysis and linear programming. Now is the time to use the game theory concepts to link together these two seemingly different types of models to widen their scopes in solving more realistic decision-making problems. The investment problem can be formulated as if the investor is playing a game against nature.

Suppose our investor has \$100,000 to allocate among the three possible investments with the unknown amounts Y_1 , Y_2 , Y_3 , respectively. That is,

$$Y_1 + Y_2 + Y_3 = 100,000$$

Notice that this condition is equivalent to the total probability condition for player I in the Game Theory.

Under these conditions, the returns are:

$$\begin{array}{llll}
 0.12Y_1 & + & 0.15Y_2 & + & 0.07Y_3 & \{\text{if Growth (G)}\} \\
 0.08Y_1 & + & 0.09Y_2 & + & 0.07Y_3 & \{\text{if Medium G}\} \\
 0.07Y_1 & + & 0.05Y_2 & + & 0.07Y_3 & \{\text{if No Change}\} \\
 0.03Y_1 & - & 0.02Y_2 & + & 0.07Y_3 & \{\text{if Low}\}
 \end{array}$$

The objective is that the smallest return (let us denote it by v value) be as large as possible.

Formulating this Decision Analysis problem as a Linear Programming problem, we have:

Max v

Subject to:

$$\begin{array}{llll}
 Y_1 & + & Y_2 & + & Y_3 & = & 100,000 \\
 0.12Y_1 & + & 0.15Y_2 & + & 0.07Y_3 & \geq & v \\
 0.08Y_1 & + & 0.09Y_2 & + & 0.07Y_3 & \geq & v \\
 0.07Y_1 & + & 0.05Y_2 & + & 0.07Y_3 & \geq & v \\
 0.03Y_1 & - & 0.02Y_2 & + & 0.07Y_3 & \geq & v
 \end{array}$$

And $Y_1, Y_2, Y_3 \geq 0$, while v is unrestricted in sign (could have negative return).

This LP formulation is similar to the problem discussed in the Game Theory section. In fact, the interpretation of this problem is that, in this situation, the investor is playing against nature (the states of economy).

Solving this problem by any LP solution algorithm, the optimal solution is $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 100,000$, and $v = \$7000$. That is, the investor must put all the money in the money market account with the accumulated return of $100,000 \times 1.07 = \$107,000$.

Note that the pay-off matrix for this problem has a saddle-point; therefore, as expected, the optimal strategy is a pure strategy. In other words, we have to invest all our money into one portfolio only.

Summery

In several situations, managers are required to make decisions in a competing situation where there are two or more parties with conflicting interests will interact and the outcome is controlled by the decision of all the parties concerned. Such problems occur frequently in Business Administration, Economics, Sociology, Political Science, and Military Training. Life is full of conflict and competition. Numerous examples involving adversaries in conflict include parlor games, military battles, political campaigns, advertising and marketing campaigns by competing business firms, and so forth. Just as people generally try to win games, people also try to "win" or achieve their interests or goals in competitive situations. However, both in games and in the real world, we generally follow a set of rules to do this. Some games, like some real situations are "winner-take-all." These games are by their nature very competitive, as only one person can win.

The Prisoner's Dilemma is one of the best-known models in game theory. It illustrates the paradoxical nature of interaction between mutually suspicious participants with opposing interests.

Self Test exercise 6

Part I: Write "True" if the statement is correct and "False" if the statement is incorrect.

1. Player is an individual or a firm or a group of firms involved in a competitive situation.
2. Game theory is the formal study of conflict and cooperation.
3. Game theoretic concepts apply whenever the actions of several agents are interdependent.

4. The concepts of game theory provide a language to formulate structure, analyze, and understand strategic scenarios.
5. A primary objective of game theory is the development of *rational criteria* for selecting a strategy.

Part II: Fill the Blank spaces with the appropriate answer.

1. A game of between persons, in which the gains of one player are same as the losses of the other player, is called _____
2. A two-person game is characterized by _____, _____, and _____
3. The Two key assumptions of game theory are: _____ and _____.

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Answer Key for self-check Exercises

Unit One: Self-Test Exercises 1

Multiple choices:

1. *D*
2. *D*
3. *E*
4. *B*

Unit Two: Self-Test Exercises 2

True/ False:

1. *False*
2. *False*
3. *True*

Multiple choices:

1. *E*
2. *C*

Unit Three: Self-Test Exercises 3

True/ False:

1. *False*
2. *False*
3. *False*
4. *False*

Unit Four: Self-Test Exercises 4

True/ False:

1. *False*
2. *False*
3. *False*

Multiple choices:

1. *C*
2. *C*
3. *A*

Unit Five: Self-Test Exercises 5

Multiple choices:

1. *F*
2. *A*
3. *C*

Unit Six: Self-Test Exercises 6

True/ False:

- | | |
|---------|---------|
| 1. True | 4. True |
| 2. True | 5. True |
| 3. True | |

Fill in the blank space

1. Two-Person Zero-Sum Game
2.
 - A. The strategies of player 1
 - B. The strategies of player 2
 - C. The payoff table
3.
 - A. Both players are *rational*.
 - B. Both players choose their strategies solely to *promote their own welfare* (no compassion for the opponent).

QUANTITATIVE ANALYSIS FOR MANAGEMENT DECISION

ASSIGNMENT

SHOW ALL THE NECESSARY STEPS

1. A manufacturer of lawn and garden equipment makes two basic types of lawn mowers: a push-type and a self-propelled model. The push-type requires 9 minutes to assemble and 2 minutes to package; the self-propelled mower requires 12 minutes to assemble and 6 minutes to package. Each type has an engine. The company has 12 hrs of assembly time available, 75 engines, and 5hrs of packing time. Profits are Birr 70 for the self-propelled models and Birr 45 for the push-type mower per unit.

Required:

A. Formulate the linear programming models for this problem.

B. Determine how many mower of each type to make in order to maximize the total profit (use both Graphic and simplex procedure).

2. A diet is to include at least 140 mgs of vitamin A and at least 145 Mgs of vitamin B. These requirements are to be obtained from two types of foods: Type 1 and Type 2. Type 1 food contains 10Mgs of vitamin A and 20mgs of vitamin B per pound. Type 2 food contains 30mgs of vitamin A and 15 mgs of vitamin B per pound. If type 1 and 2 foods cost Birr 5 and Birr 8 per pound respectively, how many pounds of each type should be purchased to satisfy the requirements at a minimum cost?

<u>Foods</u>	<u>Vitamins</u>	
	<u>A</u>	<u>B</u>
Type 1	10	20
Type 2	30	15

A. Formulate the linear programming models for this problem.

B. Determine how many mower of each type to make in order to minimize the total cost (use both Graphic and simplex procedure).

3.

Source \ Destination	D	E	F	G	Supply
A	1	5	3	4	100
B	4	2	2	5	60
C	3	1	2	4	120
demand	70	50	100	60	280

Develop the initial feasible solution for the above Transportation Problem using:

A. North West Corner Point (NWCM)

B. Least-Cost Method (LCM)

C. Vogel'S Approximation Method (VAM)

D. Once you find its initial feasible solution using NWCM, test its optimality using Steepest descent and Modified distribution and explain in detail the difference between Steepest descent and Modified distribution

3. A manager has prepared the following table, which shows the costs for various combinations of job-machine assignments:

		<i>Machine</i> (Cost in '000s)		
		A	B	C
	1	20	15	31
Job	2	17	16	33
	3	18	19	27

a. What is the optimal (minimum-cost) assignment for this problem?

b. What is the total cost for the optimum assignment?